Section 15.5: Center of Mass and Moments of Inertia

Goals:
1. To find the mass of a planar lamina using a double integral
2. To find the center of mass of a planar lamina using double integrals
3. To find moments of inertia using double integrals

Recall:
1. A thin, flat plate of some material is called a **planar lamina**.
2. Density at some spot on the lamina is measured in mass per unit of area.
3. If a region *R* corresponding to a lamina is of uniform density ρ, then the mass of this lamina is given by $m = (density)(area) = \rho A = \rho \iint_R dA = \iint_R \rho dA$.
4. If the mass of a lamina varies, we can use a double integral and a density function to find the mass of the lamina

Definition 1: Let $\rho(x, y)$ be a continuous density function on a lamina corresponding to a plane region *D*, then the mass m of the lamina is given by

$$m = \iint_D \rho(x, y) dA$$

Definition 2: The measure of the tendencies of a lamina to rotate about the *x*- and *y*- axes are called the **moments of mass** with respect to the *x*- and *y*-axes.

If we break up a region *D* into grid of equivalent rectangles and select the sample point (x_i, y_j) from the upper right corner of the inside the *ij*th rectangle, we can use it to approximate the moment of mass of that particular rectangle with respect to the *x*-axis using the following formula:

$$\text{(mass)(directed distance from x-axis)} \approx (\rho(x_i, y_j)\Delta A\ )y_j$$

Likewise, we can estimate moment of mass with respect to the *y*-axis:

$$\text{(mass)(directed distance from } y\text{-axis)} \approx (\rho(x_i, y_j)\Delta A)x_i$$

Definition 3: a) Using Riemann sums involving the above formulas, we arrive at the following definitions for **moments of mass of a lamina *R* with respect to the *x*- and *y*-axes**.

$$M_x = \iint_D y\rho(x, y)dA \text{ and } M_y = \iint_D x\rho(x, y)dA$$

b) The **center of mass of the lamina** is given by the following formulas:

$$\boxed{\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)dA} \text{ and } \boxed{\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)dA}$$

where *m* is the mass of the lamina.

Note: The **center of mass** of a planar lamina, $(\bar{x}, \bar{y})$, is the balance point of the lamina. In other words, the lamina balances horizontally when supported at its center of mass.

Definition 4: The moment of inertia of a particle of mass *m* about a line is $I = md^2$ where *d* is the distance to the line.

Using integrals, we can generalize the above definition to a lamina with density function $\rho(x, y)$.

Definition 5: The **moments of inertia** about the *x* and *y*-axes are defined by

$$I_x = \iint_D y^2 \rho(x, y) dA \text{ and } I_y = \iint_D x^2 \rho(x, y) dA$$

Note: In general, a moment of inertia plays much the same role in rotational motion that mass plays in linear motion. For example, the moment of inertia of a wheel is what makes it difficult to start or stop the rotation of the wheel, just as the mass of a car is what makes it difficult to start or stop the motion of the car.

Definition 6: The **polar moment** of inertia is $I_0 = I_x + I_y$.

Note: The polar moment of inertia is also called the **moment of inertia about the origin** (like a wheel about its axle).

Definition 7: The **radius of gyration** of a revolving mass *m* with moment of inertia *I* is defined as $\bar{\bar{r}} = \sqrt{\frac{I}{m}}$.

Notes:

1. If the entire mass of the lamina were concentrated at a distance of $\bar{\bar{r}}$ from the axis of rotation, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

2. The radius of gyration about the *y*-axis is $\boxed{\bar{\bar{x}} = \sqrt{\frac{1}{m}\iint_D x^2 \rho(x, y) dA}}$

3. The radius of gyration about the *x*-axis is $\boxed{\bar{\bar{y}} = \sqrt{\frac{1}{m}\iint_D y^2 \rho(x, y) dA}}$