Calculus II

Math 142 Fall 2008

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Exam 2

Solutions

A few integrals:

(1) $$\int \sec^3(u)\ du = \frac{1}{2}\tan(u)\sec(u) + \frac{1}{2}\ln|\sec(u)+\tan(u)| + C$$

(2) $$\int \frac{\sqrt{2au-u^2}}{u}\ du = \sqrt{2au-u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$$

(3) $$\int \frac{\sqrt{a+bu}}{u}\ du = 2\sqrt{a+bu} + \sqrt{a}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C \text{ if } a>0$$

(4) $$\int u\cos^{-1}u\ du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$$

Problem 1. *(10pts) Show how to compute* $\displaystyle\int \frac{\ln x}{x^{3/2}}\ dx$.

Solution. We use integration by parts, $\displaystyle\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$, with the table:

$f(x) = \ln x$	$g(x) = \frac{-2}{x^{1/2}}$
$f'(x) = 1/x$	$g'(x) = \frac{1}{x^{3/2}}$

Thus

$$\int \frac{\ln x}{x^{3/2}}\ dx = \frac{-2\ln x}{x^{1/2}} - \int \frac{-2}{x^{3/2}}\ dx = \frac{-2\ln x}{x^{1/2}} + 2\int \frac{1}{x^{3/2}}\ dx = \frac{-2\ln x}{x^{1/2}} + 2\frac{-2}{x^{1/2}} + C.$$

□

Problem 2. *(15pts) Show that* $\displaystyle\int_{\sqrt{5}}^{\sqrt{10}} \frac{x^3}{(\sqrt{x^2-1})^3}\ dx = 7/6.$

Solution. We use inverse trig substitution. Let $x = \sec\theta$ whence the lie is that $dx = \sec\theta\tan\theta\ d\theta$. So

$$\int_{\sqrt{5}}^{\sqrt{10}} \frac{x^3}{(\sqrt{x^2-1})^3}\ dx = \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{\sec^3\theta\sec\theta\tan\theta}{(\sqrt{\sec^2\theta-1})^3}\ d\theta = \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{\sec^4\theta\tan\theta}{(\sqrt{\tan^2\theta})^3}\ d\theta$$

$$= \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{\sec^4\theta\tan\theta}{(\tan\theta)^3}\ d\theta = \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{\sec^4\theta}{\tan^2\theta}\ d\theta$$

$$= \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{\sec^2\theta\sec^2\theta}{\tan^2\theta}\ d\theta = \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{(\tan^2\theta+1)\sec^2\theta}{\tan^2\theta}\ d\theta.$$

Now let $u = \tan\theta$, whence the fib is that $du = \sec^2\theta\ d\theta$ and we obtain:

$$\int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{(\tan^2\theta+1)\sec^2\theta}{\tan^2\theta}\ d\theta = \int_{x=\sqrt{5}}^{x=\sqrt{10}} \frac{(u^2+1)}{u^2}\ du = \int_{x=\sqrt{5}}^{x=\sqrt{10}} \left(1+\frac{1}{u^2}\right)\ du$$

$$= u - \frac{1}{u}\Bigg|_{x=\sqrt{5}}^{x=\sqrt{10}} = \tan\theta - \frac{1}{\tan\theta}\Bigg|_{x=\sqrt{5}}^{x=\sqrt{10}}.$$

Using the usual triangle, we find that $\tan\theta = \sqrt{x^2-1}$ so we have

$$\sqrt{x^2-1} - \frac{1}{\sqrt{x^2-1}}\Bigg|_{\sqrt{5}}^{\sqrt{10}} = \sqrt{(\sqrt{10})^2-1} - \frac{1}{\sqrt{(\sqrt{10})^2-1}} - \left(\sqrt{(\sqrt{5})^2-1} - \frac{1}{\sqrt{(\sqrt{5})^2-1}}\right)$$

$$= \sqrt{9} - \frac{1}{\sqrt{9}} - \left(\sqrt{4} - \frac{1}{\sqrt{4}}\right) = 3 - \frac{1}{3} - 2 + \frac{1}{2} = \frac{7}{6}.$$

Another method is to start by taking $u = x^2 - 1$, which works out nicely.w

□

Problem 3. *(15pts) Demonstrate how to compute* $\int \frac{x+1}{(x^2+1)(x-1)}\ dx$.

Solution. We do expansion by partial fractions. Write

$$\frac{x+1}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1},$$

then clear denominators to obtain the equation

$$x+1 = (Ax+B)(x-1) + C(x^2+1).$$

Setting $x=1$ yields

$$1+1 = (A+B)(1-1) + C(1^2+1) = (A+B)(0) + 2C = 2C,$$

or $C=1$. Thus

$$x+1 = (Ax+B)(x-1) + (x^2+1),$$

or

$$-x^2 + x = (Ax+B)(x-1).$$

Setting $x=0$ yields

$$0 = (A(0)+B)(0-1) = -B,$$

or $B=0$. Thus

$$-x^2 + x = (Ax+0)(x-1) = Ax^2 - Ax,$$

so $A=-1$. Now we have

$$\int \frac{x+1}{(x^2+1)(x-1)}\ dx = \int \left(\frac{-x}{x^2+1} + \frac{1}{x-1}\right)\ dx = -\int \frac{x}{x^2+1}\ dx + \int \frac{1}{x-1}\ dx$$

$$= -\frac{\ln|x^2+1|}{2} + \ln|x-1| + C$$

where I am using two facts which we know from class:

$$\int \frac{x}{x^2+a}\ dx = \frac{\ln|x^2+a|}{2} + C$$

and

$$\int \frac{1}{x+a}\ dx = \ln|x+a| + C.$$

□

Problem 4. *(20pts) Determine whether each integral is convergent or divergent. Show how to evaluate those that are convergent.*

(a–10pts) $\displaystyle\int_0^4 \frac{4}{\sqrt{x}}\,dx.$

Solution. This is an improper integral, so:

$$\int_0^4 \frac{4}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{4}{\sqrt{x}}\,dx = \lim_{b\to 0^+} 8\sqrt{x}\Big|_b^4 = \lim_{b\to 0^+}(8\sqrt{4}-8\sqrt{b}) = 16 - \lim_{b\to 0^+} 8\sqrt{b} = 16.$$

We conclude that the integral converges to 16. Note that we used that $\sqrt{x}$ is continuous from the right, and thus that $\displaystyle\lim_{b\to 0^+} 8\sqrt{b} = 8\sqrt{0} = 0$.

□

(b–10pts) $\displaystyle\int_1^\infty \frac{1+\sin^2(e^{\cos x})}{\sqrt{x}}\,dx.$

Solution. Note that $1+\sin^2(e^{\cos x}) \geq 1$ for all $x \geq 1$ since $\sin^2(e^{\cos x}) = (\sin(e^{\cos x}))^2 \geq 0$ for all $x \geq 1$. Thus

$$0 \leq \frac{1}{\sqrt{x}} \leq \frac{1+\sin^2(e^{\cos x})}{\sqrt{x}}$$

for all $x \geq 1$. This fact, along with the observation that $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges by the P-test, implies by the Comparison Test that $\displaystyle\int_1^\infty \frac{1+\sin^2(e^{\cos x})}{\sqrt{x}}\,dx$ diverges. □

Problem 5. *(15pts) Use the trapezoidal rule to estimate the net distance traveled by a snail from $t = 0$ to $t = 1$ if the snail has velocity $v(t) = e^{t^3}$ in inches per hour at time t. Your estimate should be have error at most $\frac{e}{20}$.*

Solution. Note that net change is the integral of rate. So we want to estimate $\displaystyle\int_0^1 e^{t^3}\,dt \approx T_n$.

We know that $|E_{T_n}| \leq \dfrac{(1-0)^3K}{12n^2}$ where K is such that $K \geq |f''(x)|$ on $[0,1]$. Here $f'(x) = 3t^2e^{t^3}$ and $f''(x) = 6te^{t^3} + 9t^4e^{t^3}$. Of course, $6te^{t^3} + 9t^4e^{t^3} \geq 0$ on $[0,1]$ since each of the component functions is positive. Also, $6t$, $9t^4$, and e^{t^3} are increasing on $[0,1]$ (simply observe that each of their derivatives 6, $32t^3$, and $3t^2e^{t^3}$ are positive on $[0,1]$), so that the maximum value of $f''(t)$ on $[0,1]$ occurs at $t = 1$, that is $6te^{t^3} + 9t^4e^{t^3} \leq 6e + 9e = 15e$ on $[0,1]$. Thus we may take $K = 15e$, and conclude that $|E_{T_n}| \leq \dfrac{15e}{12n^2} = \dfrac{5e}{4n^2}$. In order that $|E_{T_n}| \leq \dfrac{e}{20}$ we need n such that $\dfrac{5e}{4n^2} \leq \dfrac{e}{20}$, that is, such that $n^2 \geq \dfrac{5e20}{4e} = 25$. Here $n = 5$ does the trick. Now we have

$$T_5 = \frac{\Delta x}{2}\left(v(t_0) + 2v(t_1) + 2v(t_2) + 2v(t_3) + 2v(t_4) + v(t_5)\right)$$

with $\Delta x = \dfrac{1}{5}$, and

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 1/5 \\ x_2 &= 2/5 \\ x_3 &= 3/5 \\ x_4 &= 4/5 \\ x_5 &= 5/5 \end{aligned}$$

So

$$\int_0^1 e^{t^3}\,dt \approx T_5 = \frac{1}{10}\left(e^{0^3} + 2e^{(1/5)^2} + 2e^{(2/5)^2} + 2e^{(3/5)^2} + 2e^{(4/5)^2} + e^{(5/5)^2}\right)$$

inches with error at most $\dfrac{e}{20}$. □