

# Taylor series derivation

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### Deriving Taylor Series from a Basic List

The list below provides the power series for several elementary functions with the corresponding intervals of convergence.

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \dots + (-1)^n(x - 1)^n + \dots$	$0 < x < 2$
$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots$	$-1 < x < 1$
$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots + \frac{(-1)^{n-1}(x - 1)^n}{n} + \dots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \dots$	$-1 \leq x \leq 1$
$(1 + x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \dots$	$-1 < x < 1^*$

\* The convergence at  $x = \pm 1$  depends on the value of  $k$ .

Note that the binomial series is valid for noninteger values of  $k$ . Also, when  $k$  is a positive integer, the binomial series reduces to a simple binomial expansion.

#### EXAMPLE 6 Deriving a Power Series from a Basic List

Find the power series for

$$f(x) = \cos \sqrt{x}.$$

**Solution** Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

you can replace  $x$  by

$$\sqrt{x}$$

to obtain the series

$$\cos \sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \dots$$

This series converges for all  $x$  in the domain of  $\cos \sqrt{x}$ —that is, for  $x \geq 0$ . ■