Problem Set 9

Section 16.5:

1) The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$, $y = \sqrt{4 - x^2}$ together with the parts of the positive x-axis that joins them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

Solution:

Best to do in polar coordinates.

$\rho = kr$

$$m = \int_0^{\pi} \int_1^2 kr^2 dr d\theta = 7/3\pi k$$

$M_y = \int\int x\rho dA = 0$ by symmetry

$$M_x = \int\int_D y\rho dA = \int_0^{\pi}\int_1^2 (r\sin\theta)(kr) r dr d\theta = 15/2k$$

Thus $(\overline{x}, \overline{y}) = (0, 45/(14\pi)$.

2) Consider the lamina bounded by $y = 1 - x^2$ and $y = 0$ with density $\rho(x, y) = ky$. Find the moments of inertia I_x, I_y, I_0.

Solution

$$I_x = \int\int_D y^2 \rho dA = \int_{-1}^{1}\int_0^{1-x^2} y^2 ky dy dx = (64/215)k$$

$$I_y = \int\int_D x^2 \rho dA = \int_{-1}^{1}\int_0^{1-x^2} kx^2 dydx = (8/105)k$$

$$I_0 = I_x + I_y = (\pi/12)k.$$

Section 16.6:

3) Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

Solution

Since $f_x = y$ and $f_y = x$

$$A(S) = \int\int_D \sqrt{1 + x^2 + y^2} dA = \int_0^{2\pi}\int_0^1 \sqrt{1 + r^2} = \frac{2\pi}{3}(2\sqrt{2} - 1)$$

4) Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$.

Solution:

Surface lies above the disk $x^2 + z^2$ in the xz plane.

$$A(S) = \int\int_D \sqrt{f_x^2 + f_z^2} dA = \int\int \sqrt{4x^2 + 4y^2 + 1} da$$

Converting to polar coords get

$$\int_0^{2\pi}\int_0^5 \sqrt{4r^2 + 1} r dr d\theta = \pi/8(101\sqrt{101} - 1).$$

Section 16.7:

5) Find $\int\int\int_D xydV$ where E is the region bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$.

Solution:

E is the solid region above the Type I region in the plane between the curves $y = x^2$ $y = \sqrt{x}$ and below the plane $z = x + y$.

Hence we get

$$\int\int\int xydV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xydzdvdx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^y + xy^2)dydx$$

This equals $3/28$.

Section 16.8:

6) Find the volume of the solid that lies between the paraboloid $z = x^2+y^2$ and the sphere $x^2+y^2+z^2 = 2$.

Solution:

In cylindrical coordinates E is bounded below by the cone $z = r$ and above by the sphere $z^2 + r^2 = 2$.

The cone and the sphere intersecct when $r = 1$ so

$$E = (r, \theta.z)|0 \le \theta \le 2\pi, 0 \le r \le 1, r \le z \le \sqrt{2 - r^2}$$

.

This gives volume

$$\int\int\int_E dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} rdzdrd\theta = 4/3\pi(\sqrt{2}-1)$$

7) Compute $\int\int\int_B (x^2+y^2+z^2)^2 dV$ where B is the ball with center the origin and radius 5.

Solution:

$$\int\int\int_B (x^2+y^2+z^2)^2 dV = \int_0^{\pi}\int_0^{2\pi}\int_0^5 (\rho^2)^2\rho^2 \sin\phi d\rho d\theta d\phi$$

This is approximately 140.

Section 16.9:

8) Use the transformation $u = xy$, $v = xy^2$ to compute $\int\int_R y^2 dA$ where R is the region bounded by the curves $xy = 1$, $xy = 2$, $xy^2 = 1$, $xy^2 = 2$.

Solution:

Have $y = u/v$ and $x = u^2/v$ so

$$\frac{\partial(x,y)}{\partial(u,v)} = 1/v$$

and R is a square with vertices $(1,1), (1,2), (2,2), (1,2)$ so that

$$\int\int_R y^2 dA = \int_1^2\int_1^2 \frac{v^2}{u^2}\left(\frac{1}{v}\right) dudv = 3/4$$