Math 314 Lecture #23
§15.5: Applications of Double Integrals

Outcome A: Use double integration to recover mass from the density.

Suppose a lamina occupies a region D of the xy-plane, and is density (mass per unit area) is given by a function $\rho(x,y)$ continuous on D.

The density of the lamina at a point (x,y) in D is given by

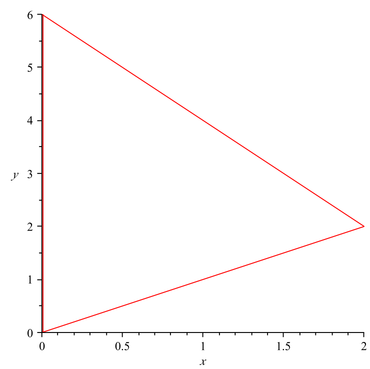
$$\lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

where Δm is the mass of the rectangles R containing the point (x,y) whose area ΔA is shrinking to 0.

The mass m of the lamina is recovered from its density through the double integral:

$$m = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\ \Delta A = \iint_D \rho(x,y)\ dA.$$

Example. Find the mass of the lamina whose shape is the triangular region D enclosed by the lines $x=0$, $y=x$, and $2x+y=6$, and whose density is $\rho(x,y)=x+y$. Here is a picture of the region D.



The region D is of both types, but is easier to render it as of type I, namely

$$D = \{(x,y) : 0 \le x \le 2, x \le y \le 6-2x\}.$$

The mass of the lamina is

$$\begin{aligned}\iint_D \rho(x,y)\ dA &= \int_0^2 \int_x^{6-2x} (x+y)\ dydx = \int_0^2 \left[xy + \frac{y^2}{2}\right]_{y=x}^{y=6-2x} dx \\ &= \int_0^2 \left[x(6-2x) + \frac{(6-2x)^2}{2} - x^2 - \frac{x^2}{2}\right] dx \\ &= \int_0^2 \left[6x - \frac{7x^2}{2} + \frac{36-24x+4x^2}{2}\right] dx \\ &= \int_0^2 \left[18 - 6x - \frac{3x^2}{2}\right] dx = \left[18x - 3x^2 - \frac{x^3}{2}\right]_0^2 = 36 - 12 - \frac{8}{2} = 20.\end{aligned}$$

Outcome B: Use double integration to compute moments and center of mass of lamina.

Recall that the moment of a particle about an axis is the product of its mass and its directed distance from the axis.

The **moment** of a lamina occupying a region D with density $\rho(x, y)$ **about the** x-axis is

$$M_x = \iint_D y\rho(x, y)\ dA.$$

The **moment** of the lamina **about the** y-axis is

$$M_y = \iint_D x\rho(x, y)\ dA.$$

The **center of mass** of the lamina is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}$$

where m is the mass of the lamina; the center of mass is the point on which the lamina would balance perfectly.

When the density is uniform, i.e., $\rho(x, y)$ is a constant, the center of mass is the geometric center. Where is the geometric center of North America?

Example (continued). The moment about the x-axis of the lamina occupying the region D enclosed by $x = 0$, $y = x$, and $2x + y = 6$, with density $\rho(x, y) = x + y$ is

$$\begin{aligned}
M_x = \iint_D y\rho(x, y)\ dA &= \int_0^2 \int_x^{6-2x} y(x + y)\ dA = \int_0^2 \int_x^{6-2x} (xy + y^2)\ dA \\
&= \int_0^2 \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{y=x}^{y=6-2x} dx \\
&= \int_0^2 \left[\frac{x(6-2x)^2}{2} + \frac{(6-2x)^3}{3} - \frac{x^3}{2} - \frac{x^3}{3}\right] dx \\
&= \int_0^2 \left[\frac{x(36 - 24x + 4x^2)}{2} - \frac{5x^3}{6} + \frac{(6-2x)^3}{3}\right] dx \\
&= \int_0^2 \left[18x - 12x^2 + \frac{7x^3}{6} + \frac{(6-2x)^3}{3}\right] dx \\
&= \left[9x^2 - 4x^3 + \frac{7x^4}{24} - \frac{(6-2x)^4}{24}\right]_0^2 \\
&= 36 - 32 + \frac{7(16)}{24} - \frac{16}{24} + \frac{6^4}{24} = 62.
\end{aligned}$$

The y-component of the center of mass is

$$\bar{y} = \frac{M_x}{m} = \frac{62}{20} = \frac{31}{10}.$$

The moment of the lamina about the y-axis is

$$
\begin{aligned}
M_y = \iint_D x\rho(x,y)\ dA &= \int_0^2 \int_x^{6-2x} (x^2 + xy)\ dydx \\
&= \int_0^2 \left[x^2 y + \frac{xy^2}{2} \right]_{y=x}^{y=6-2x} dx \\
&= \int_0^2 \left[x^2(6-2x) + \frac{x(6-2x)^2}{2} - x^3 - \frac{x^3}{2} \right] dx \\
&= \int_0^2 \left[6x^2 - 2x^3 + \frac{x(36 - 24x + 4x^2)}{2} - \frac{3x^3}{2} \right] dx \\
&= \int_0^2 \left[18x - 6x^2 - \frac{3x^3}{2} \right] dx \\
&= \left[9x^2 - 2x^3 - \frac{3x^4}{8} \right]_0^2 \\
&= 36 - 16 - 6 = 14.
\end{aligned}
$$

The x-component of the center of mass is

$$\bar{x} = \frac{M_y}{m} = \frac{14}{20} = \frac{7}{10}.$$

Recall that the momenta of inertia, or the second moment, of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis.

The **moment of inertia** of a lamina occupying a region D with density $\rho(x,y)$ **about the** x-axis is

$$I_x = \iint_D y^2 \rho(x,y)\ dA,$$

the momenta of inertia **about the** y-axis is

$$I_y = \iint_D x^2 \rho(x,y)\ dA,$$

and the momenta of inertia **about the origin** is

$$I_0 = \iint_D (x^2 + y^2)\rho(x,y)\ dA.$$

Notice that $I_0 = I_x + I_y$.

Example (continued). For the lamina in the previous examples,

$$I_x = \frac{1104}{5}, \quad I_y = \frac{72}{5}, \quad I_0 = \frac{1176}{5}.$$