Math 254 ~ Multiple Integration

14.1 – Iterated Integrals and Area in the Plane

Iterated Integrals

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

Use *partial integration with respect to y* to compute the inner integral (treating *x* as a constant.)

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

Use *partial integration with respect to x* to compute the inner integral (treating *y* as a constant.)

Note: In each of these iterated integrals, the inside limits of integration can be variable with respect to the outer variable of integration. However, the outside limits of integration *must be* constant with respect to both variables of integration.

Definition – A **vertically simple region**, *R*, is a region in the *xy*-plane that lies between the graphs of two continuous functions of *x*, that is,

$$R = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$.

Definition – A **horizontally simple region**, *R*, is a region in the *xy*-plane that lies between the graphs of two continuous functions of *y*, that is,

$$R = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous on $[c,d]$.

Area of a Region in the Plane

If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy dx$$

If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx dy$$

Note: If a region is both vertically simple and horizontally simple, then the iterated integral can be ordered in either way, that is, *dxdy* or *dydx*.

Note: The order of integration can greatly affect the difficulty of the integration!

Note: To change the order of integration, it helps to sketch the region.

14.2 – Double Integrals and Volume

Definition – Double Integral

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is given by

$$\iint_R f(x,y)\,dA = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

Volume of a Solid Region

If f is integrable over a plane region R and $f(x,y) \ge 0$ for all (x,y) in R, then the volume of the solid region that lies above R and below the graph of f is defined as

$$V = \iint_R f(x,y)\,dA$$

Theorem – Properties of Double Integrals
Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1) $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$

2) $\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

3) $\iint_R f(x,y)\,dA \ge 0$, if $f(x,y) \ge 0$ for all (x,y) in R

4) $\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA$, if $f(x,y) \ge g(x,y)$ for all (x,y) in R

5) $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$, where R is the union of two non-overlapping subregions R_1 and R_2.

Theorem – Fubini's Theorem

Let f be continuous on a plane region R.

1) If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2) If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

14.3 – Change of Variables: Polar Coordinates

Definition – A **polar sector** is a region R defined by

$$R = \{(r,\theta) \mid r_1 \le r \le r_2,\ \ \theta_1 \le \theta \le \theta_2\},\ \text{where } r_1, r_2, \theta_1, \theta_2 \text{ are constants.}$$

Definition – An ***r*-simple region**, R, is a region in the $r\theta$-plane that lies between the graphs of two continuous functions of θ, that is,

$$R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ \ 0 \le g_1(\theta) \le r \le g_2(\theta)\}$$

where g_1 and g_2 are continuous on $[\alpha, \beta]$.

Definition – An **θ-simple region**, R, is a region in the $r\theta$-plane that lies between the graphs of two continuous functions of r, that is,

$$R = \{(r,\theta) \mid r_1 \le r \le r_2,\ \ h_1(r) \le \theta \le h_2(r)\}$$

where h_1 and h_2 are continuous on $[r_1, r_2]$.

Theorem – Change of Variables to Polar Coordinates

Let R be a region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Note: There is an extra factor of r in the polar form!!! $rdrd\theta$!!!

14.4 – Center of Mass and Moments of Inertia

Definition of Mass of a Planar Lamina of Variable Density

If ρ is a continuous density function on the lamina corresponding to a plane region *R*, then the mass *m* of the lamina is given by

$$m = \iint_R \rho(x, y)dA$$

Note: If the lamina has a constant density, then $\rho(x, y)$ is a constant function, call it simply the constant ρ, and then the mass is simply

$$m = \iint_R \rho dA = \rho \iint_R dA = \rho A.$$

Moments and Center of Mass of a Variable Density Planar Lamina

Let ρ be a continuous density function on the planar lamina *R*. The **moment of mass** with respect to the *x*- and *y*-axes are

$$M_x = \iint_R y\rho(x, y)dA \quad \text{and} \quad M_y = \iint_R x\rho(x, y)dA.$$

If *m* is the mass of the lamina, then the **center of mass** is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right).$$

If *R* represents a simple plane region rather than a lamina, the point $(\overline{x}, \overline{y})$ is called the **centroid** of the region.

14.5 – Surface Area

Definition of Surface Area

If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the **area of the surface *S*** given by $z = f(x, y)$ over R is given by

$$\text{Surface Area} = \iint_S dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA .$$

14.6 – Triple Integrals and Applications

Definition of Triple Integral

If f is continuous over a bounded solid region Q, then the **triple integral of *f* over *Q*** is defined as

$$\iiint_Q f(x, y, z)dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV .$$

Theorem – Evaluation by Iterated Integrals

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x, y) \le z \le g_2(x, y)$$

where h_1, h_2, g_1, g_2 are continuous functions. Then,

$$\iiint_Q f(x, y, z)dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dzdydx .$$

14.7 – Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

Given Q is a solid region whose projection onto the xy-plane is the region R. Suppose that

$$Q = \{(x,y,z) \mid (x,y) \text{ is in } R,\ h_1(x,y) \le z \le h_2(x,y)\}$$

and

$$R = \{(r,\theta) \mid \theta_1 \le \theta \le \theta_2, g_1(\theta) \le r \le g_2(\theta)\}.$$

If f is a continuous function on the solid Q, you can write the triple integral of f over Q as

$$\iiint_Q f(x,y,z)\,dV = \iint_R \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz \right] dA$$

where the double integral over R is evaluated in polar coordinates.

So, if R is r-simple, the iterated form of the triple integral in cylindrical form is

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r\cos\theta, r\sin\theta)}^{h_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

Spherical Coordinates

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$$\iiint_Q f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Note: These formulas can be modified for different orders of integration and generalized to include regions with variable boundaries.

14.8 – Change of Variables: Jacobians

Definition of the Jacobian

If $x = g(u,v)$ and $y = h(u,v)$, then the **Jacobian** of *x* and *y* with respect to *u* and *v*, denoted by $\partial(x,y)/\partial(u,v)$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Theorem – Change of Variables for Double Integrals

Let *R* and *S* be regions in the *xy*- and *uv*-planes (respectively) that are related by the equations $x = g(u,v)$ and $y = h(u,v)$ such that each point in *R* is the image of a unique point in *S*. If f is continuous on *R*, *g* and *h* have continuous partial derivatives on *S*, and $\partial(x,y)/\partial(u,v)$ is nonzero on *S*, then

$$\iint_R f(x,y)\,dxdy = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$