Math 3321 – Lecture 16 notes

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\dfrac{1}{s}, \qquad s > 0$
$e^{\alpha x}$	$\dfrac{1}{s-\alpha}, \qquad s > \alpha$
$\cos\, \beta x$	$\dfrac{s}{s^2+\beta^2}, \qquad s > 0$
$\sin\, \beta x$	$\dfrac{\beta}{s^2+\beta^2}, \qquad s > 0$
$e^{\alpha x} \cos\, \beta x$	$\dfrac{s-\alpha}{(s-\alpha)^2+\beta^2}, \qquad s > \alpha$
$e^{\alpha x} \sin\, \beta x$	$\dfrac{\beta}{(s-\alpha)^2+\beta^2}, \qquad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \qquad s > 0$
$x^n\, e^{rx}, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-r)^{n+1}}, \qquad s > r$
$x\, \cos\, \beta x$	$\dfrac{s^2-\beta^2}{(s^2+\beta^2)^2}, \qquad s > 0$
$x\, \sin\, \beta x$	$\dfrac{2\beta s}{(s^2+\beta^2)^2}, \qquad s > 0$

Section 4.6 Application to Initial-Value Problems

Given $y' + ay = f(x), y(0) = \alpha$ or $y'' + ay' + by = f(x), y(0) = \alpha, y'(0) = \beta$ and $f(x)$ a piecewise function, we will solve like this:

1. Express $f(x)$ in terms of a step function
2. Calculate $\mathcal{L}[y(x)] = Y(s)$ using the properties
   $\mathcal{L}\left[y'(x)\right] = -y(0) + s\mathcal{L}\left[y(x)\right]$ *and* $\mathcal{L}\left[y''(x)\right] = -y'(0) - sy(0) + s^2\mathcal{L}\left[y(x)\right]$
3. Take $\mathcal{L}^{-1}[F(s)]$ to find $y(x)$

Examples:

1. Find the solution of the initial-value problem $y' + 2y = f(x)$; $y(0) = 4$ where

$$f(x) = \begin{cases} x, & 0 \le x < 3 \\ 5, & x \ge 3 \end{cases}$$

$y' + 2y = f(x)$

No characteristic polynomial if Laplace!

$f(x) = x + (5 - x)u(x-3)$

$y' + 2y = x + (5-x)u(x-3)$

$g(x-3) = 5 - x$

$g((x+3)-3) = 5 - (x+3)$

$g(x) = 2 - x$

$$\mathcal{L}\left[y' + 2y = x + g(x-3)\,u(x-3)\right]$$

$$-y(0) + sY + 2Y = \frac{1}{s^2} + e^{-3x}\mathcal{L}[g(x)]$$

$\overset{\shortparallel}{4}$

$$\mathcal{L}[g(x)] = \mathcal{L}[2-x]$$

$$= \frac{2}{s} - \frac{1}{s^2}$$

$$-4 + sY + 2Y = \frac{1}{s^2} + e^{-3x}\left[\frac{2}{s} - \frac{1}{s^2}\right]$$

$$(s+2)Y = \frac{1}{s^2} + e^{-3x}\left(\frac{2}{s}\right) - e^{-3x}\left(\frac{1}{s^2}\right) + 4$$

$$Y = \frac{1}{s^2(s+2)} + e^{-3x}\left(\frac{2}{s(s+2)}\right) - e^{-3x}\left(\frac{1}{s^2(s+2)}\right) + \frac{4}{s+2}$$

$$Y = \left(\frac{1}{s^2(s+2)} + \frac{4}{s+2}\right) + e^{-3x}\left(\frac{2}{s(s+2)} - \frac{1}{s^2(s+2)}\right)$$

$$Y = \left(\frac{4s^2+1}{s^2(s+2)}\right) + e^{-3x}\left(\frac{2s-1}{s^2(s+2)}\right)$$

$$\frac{4s^2+1}{s^2(s+2)} = -\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{17}{4}\left(\frac{1}{s+2}\right)$$

$$\frac{2s-1}{s^2(s+2)} = \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{1}{s+2}\right) + \frac{3}{4}\left(\frac{1}{s}\right)$$

$$Y = \left[-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{17}{4}\left(\frac{1}{s+2}\right)\right] + e^{-3x}\left[\frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{1}{s+2}\right) + \frac{3}{4}\left(\frac{1}{s}\right)\right]$$

$$y = -\frac{1}{4}(1) + \frac{1}{2}x + \frac{17}{4}e^{-2x} + h(x-c)u(x-c)$$

$$e^{-3x}\left[\frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{1}{s+2}\right) + \frac{3}{4}\left(\frac{1}{s}\right)\right]$$

$$c = 3$$

$$\mathcal{L}[h(x)] = \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{1}{s+2}\right) + \frac{3}{4}\left(\frac{1}{s}\right)$$

$$h(x) = \frac{1}{2}x - \frac{3}{4}e^{-2x} + \frac{3}{4}$$

$$h(x-5) = \frac{1}{2}(x-3) - \frac{3}{4}e^{-2(x-3)} + \frac{3}{4}$$

$$y = -\frac{1}{4} + \frac{1}{2}x + \frac{17}{4}e^{-2x} + \left[\frac{1}{2}(x-3) - \frac{3}{4}e^{-2(x-3)} + \frac{3}{4}\right]u(x-3)$$

$0 \le x < 3: \; y = -\frac{1}{4} + \frac{1}{2}x + \frac{17}{4}e^{-2x}$

$x \ge 3: \; y = -\frac{1}{4} + \frac{1}{2}x + \frac{17}{4}e^{-2x} + \frac{1}{2}(x-3) - \frac{3}{4}e^{-2(x-3)} + \frac{3}{4}$

$= -1 + x + \frac{17}{4}e^{-2x} - \frac{3}{4}e^{-2(x-3)}$

$$y = \begin{cases} -\frac{1}{4} + \frac{1}{2}x + \frac{17}{4}e^{-2x} & 0 \le x < 3 \\ -1 + x + \frac{17}{4}e^{-2x} - \frac{3}{4}e^{-2(x-3)} & x \ge 3 \end{cases}$$

Pop 09: CDCAC

Quiz 6 # 11

$$y^{(4)} + 18y'' + 81y = \text{- — - .}$$

$$\lambda^4 + 18\lambda^2 + 81 = 0$$

$$(\lambda^2 + 9)^2 = 0$$

$$\lambda = \pm 3i \quad (\text{repeated})$$

$\cos 3x, \; \sin 3x, \; x\cos 3x, \; x \sin 3x$

2. Solve $y'' + 2y' + y = f(x)$, $y(0) = y'(0) = 0$ where $f(x) = \begin{cases} 1 & 0 \le x < 2 \\ x - 1 & x \ge 2 \end{cases}$

3. Solve $y'' + 2y' + y = f(x)$, $y(0) = 3$, $y'(0) = -1$ where $f(x) = \begin{cases} e^x & 0 \le x < 1 \\ e^x - 1 & x \ge 1 \end{cases}$