Quiz 5 #1

$$y'' + 9y = -5\tan 3x$$

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm 3i \rightarrow y_1 = \cos 3x, \quad y_2 = \sin 3x$$

$$y_1' = -3\sin 3x \qquad y_2' = 3\cos 3x$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x$$

$$= 3$$

$$u = \int -\frac{y_2}{W(x)} f(x)\,dx = \int -\frac{\sin 3x}{3}(-5\tan 3x)\,dx$$

$$= \frac{5}{3} \int \sin 3x \tan 3x \, dx$$

$$= \frac{5}{3} \int \frac{\sin^2 3x}{\cos 3x} \, dx$$

$$= \frac{5}{3} \int \frac{1 - \cos^2 3x}{\cos 3x} \, dx$$

$$= \frac{5}{3} \left[\int \frac{1}{\cos 3x} \, dx - \int \cos 3x \, dx \right]$$

$$= \frac{5}{3} \left[\int \sec 3x \, dx - \int \cos 3x \, dx \right]$$

$$= \frac{5}{9} \left[\ln \left| \sec 3x + \tan 3x \right| - \sin 3x \right]$$

Quiz 6 1&2

SHM: $\boxed{y = A\sin(\omega t + \phi_0)}$

$y(0) = 1 \rightarrow 1 = A\sin\phi_0 \rightarrow A = \frac{1}{\sin\theta_0}$

$y'(0) = 8 \rightarrow A\omega\cos(\phi_0) = 8$

$\omega\cot(\phi_0) = 8$

Quiz 5 #5: $y'' + 16y = \underbrace{3\cos 5x + 4\sin 5x}$

$\rightarrow z = A\cos 5x + B\sin 5x$

$\lambda^2 + 16 = 0$

$\lambda = \pm 4i \rightarrow y_1 = \cos 4x,\ y_2 = \sin 4x$

Quiz 6 # 9

$$y = \underbrace{2e^{4x}}_{\downarrow} - \underbrace{3\sin 4x}_{\downarrow} + \underbrace{2x}_{\downarrow}$$

$$Ae^{4x} \qquad B\sin 4x + C\cos 4x \qquad Dxe^{0x} + Ee^{0x}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$(\lambda - 4) \qquad \lambda = \pm 4i \qquad (\lambda - 0)^2$$

$$\downarrow \qquad\qquad \downarrow$$

$$(\lambda - 4i)(\lambda + 4i) \qquad \lambda^2$$

Char poly: $(\lambda - 4)(\lambda - 4i)(\lambda + 4i)\lambda^2 = 0$

Math 3321 – Lecture 15 notes

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\dfrac{1}{s}, \qquad s > 0$
$e^{\alpha x}$	$\dfrac{1}{s-\alpha}, \qquad s > \alpha$
$\cos\,\beta x$	$\dfrac{s}{s^2+\beta^2}, \qquad s > 0$
$\sin\,\beta x$	$\dfrac{\beta}{s^2+\beta^2}, \qquad s > 0$
$e^{\alpha x}\cos\,\beta x$	$\dfrac{s-\alpha}{(s-\alpha)^2+\beta^2}, \qquad s > \alpha$
$e^{\alpha x}\sin\,\beta x$	$\dfrac{\beta}{(s-\alpha)^2+\beta^2}, \qquad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}, \qquad s > 0$
$x^n\,e^{rx}, \quad n = 1, 2, \ldots$	$\dfrac{n!}{(s-r)^{n+1}}, \qquad s > r$
$x\,\cos\,\beta x$	$\dfrac{s^2-\beta^2}{(s^2+\beta^2)^2}, \qquad s > 0$
$x\,\sin\,\beta x$	$\dfrac{2\beta s}{(s^2+\beta^2)^2}, \qquad s > 0$

And

$$\mathcal{L}\left[f(x)\right]=\int_0^\infty e^{-sx}f(x)dx$$

$$\mathcal{L}\left[y'(x)\right]=-y(0)+s\mathcal{L}\left[y(x)\right]$$

$$\mathcal{L}\left[y''(x)\right]=-y'(0)-sy(0)+s^2\mathcal{L}\left[y(x)\right]$$

In general:

If $y, y', y'', \ldots, y^{(n-1)}$ are of exponential order λ, then $\mathcal{L}\left[y^{(n)}(x)\right]$ exists for $s>\lambda$ and

$$\mathcal{L}\left[y^{(n)}(x)\right]=s^n\mathcal{L}[y(x)]-s^{n-1}y(0)-s^{n-2}y'(0)-\ldots-y^{(n-1)}(0)$$

Using inverse Laplace Transforms to solve initial value problems:

Example: $y' + 2y = 3e^{x}, \ y(0) = 4$

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 3\mathcal{L}[e^{x}]$$

$$-y(0) + s\mathcal{L}[y] + 2\mathcal{L}[y] = 3\left(\frac{1}{s-1}\right)$$

$$-4 + sY + 2Y = 3\left(\frac{1}{s-1}\right)$$

$$(s+2)Y = \frac{3}{s-1} + 4$$

$$Y = \frac{3}{(s-1)(s+2)} + \frac{4}{s+2}$$

$\downarrow$ PFD

$$Y = \frac{1}{s-1} - \frac{1}{s+2} + \frac{4}{s+2}$$

$$\mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 3\,\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

$$\boxed{y = e^{x} + 3e^{-2x}}$$

Find the Laplace transform of the solution of the initial-value problem:
$y'' - y' - 2y = \sin(2x)$, $y(0) = 1$, $y'(0) = 2$. Then find the solution of the problem.

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = \mathcal{L}[\sin 2x]$$

$$(-y'(0) - s\,y(0) + s^2 Y) - (-y(0) + sY) - 2Y = \frac{2}{s^2+4}$$

$$-2 - s + s^2 Y + 1 + sY - 2Y = \frac{2}{s^2+4}$$

$$s^2 Y + sY - 2Y = \frac{2}{s^2+4} + s + 1$$

$$(s^2 + s - 2)Y = \frac{2 + s(s^2+4) + s^2 + 4}{s^2+4}$$

$$Y = \frac{s^3 + s^2 + 4s + 6}{(s^2+4)(s^2+s-2)}$$

$\downarrow$ PFD

$$Y = \frac{1}{20}\left(\frac{s}{s^2+4}\right) - \frac{3}{20}\left(\frac{2}{s^2+4}\right) + \frac{13}{12}\left(\frac{1}{s-2}\right) - \frac{2}{15}\left(\frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}[Y] = \frac{1}{20}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{3}{20}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \frac{13}{12}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \frac{2}{15}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$\boxed{y = \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x + \frac{13}{12}e^{2x} - \frac{2}{15}e^{-x}}$$

Sections 4.4-4.5. Discontinuous Functions

Def. Let the function $f = f(x)$ be defined on an interval *I* and continuous except at a point $c \in I$.

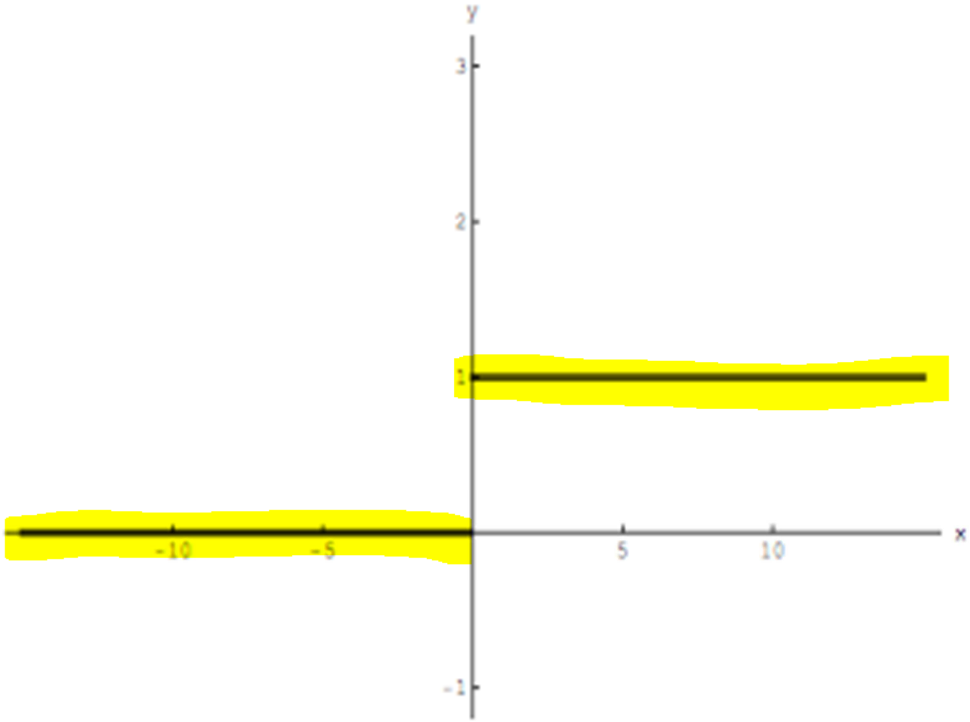
If $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ exist but $\lim_{x\to c-} f(x) \neq \lim_{x\to c+} f(x)$ then *f* is said to have **jump** discontinuity at *c*.

Def. A function *f* defined on an interval *I* is piecewise continuous on *I* if it is continuous on *I* except for at most a finite number of points $c_1, c_2, \ldots, c_n$ of *I* at which it has jump discontinuities.

Theorem: If the function *f* is piecewise continuous on $[0, \infty)$, and of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)]$ exists for $s > \lambda$.

Unit Step Functions:

The Heaviside function: $u(x)=\begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases}$



Unit Step Functions:

Let $c > 0$, The function $u_c(x) = u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$ is called a *unit step function*.

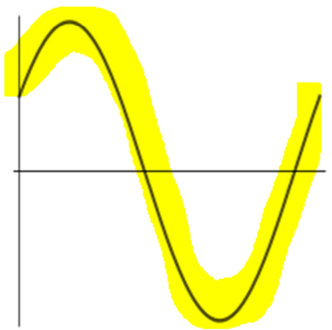
Laplace Transform of a unit step function: $\mathcal{L}[u(x-c)] = e^{-cs}\dfrac{1}{s}, \quad s > 0$

Translation of a function: If f is defined on $[0, \infty)$ and $c > 0$, then the function

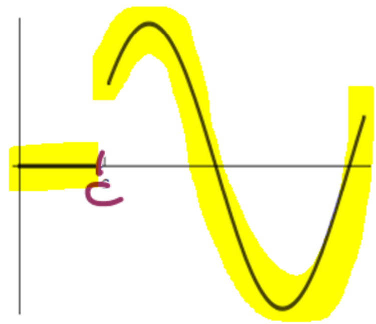
$$f(x-c)u(x-c) = \begin{cases} 0 & x < c \\ f(x-c) & x \geq c \end{cases}$$ is a translation of f.

for forced harmonic motion: if f(x) is translated, force being delayed

$f(x)$



$f(x-c)u(x-c)$



Laplace Transform of a Translated Function: Suppose that $\mathcal{L}[f(x)] = F(s)$. Then, for any positive number c:

1. $\boxed{\mathcal{L}[f(x-c)u(x-c)]} = \boxed{e^{-cs}F(s)}$ $= e^{-cs} \cdot \mathcal{L}[f(x)]$
2. $\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c)$

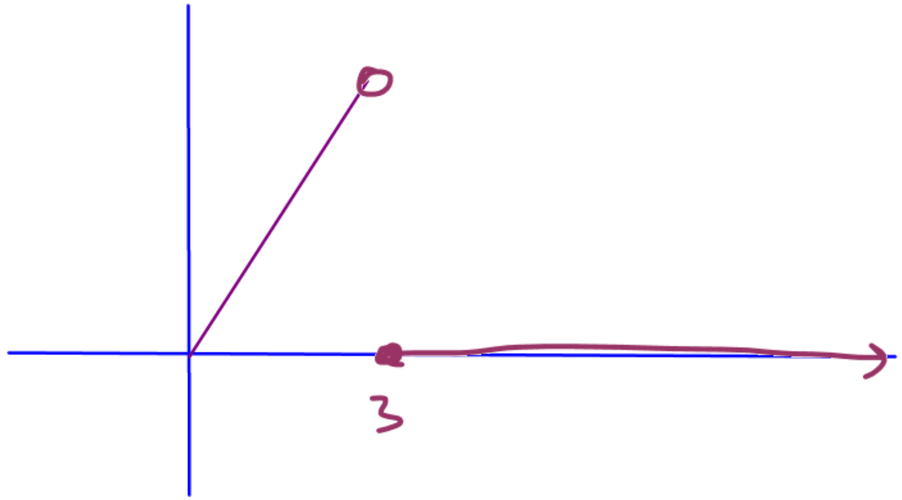
Examples:

1. Express f in terms of a step function:

$f_1(x)$

a. $f(x)=\begin{cases} 2x & 0 \le x < 3 \\ 0 & x \ge 3 \end{cases}$ c

$f_2(x)$



$$\boxed{f(x) = f_1(x) + (f_2(x) - f_1(x))\,u(x-c)}$$

Why? if $0 \le x < c$: $u(x-c) = 0$

$f(x) = f_1(x)$

if $x \ge c$: $u(x-c) = 1$

$f(x) = f_1(x) + f_2(x) - f_1(x) = f_2(x)$

$$f(x) = 2x + (0 - 2x)\,u(x-3)$$

$$f(x) = 2x - 2x\,u(x-3)$$

What is $\mathcal{L}[f(x)]$?

$2x\,u(x-3)$ needs to look like

$$g(x-3)\,u(x-3)$$

$$2x = g(x-3)$$

$$f(x) = 2x - g(x-3)\,u(x-3)$$

$$\mathcal{L}[g(x-c)u(x-c)] = e^{-cs}G(s)$$

$$\mathcal{L}[f(x)] = 2\,\mathcal{L}[x] - \mathcal{L}[g(x-3)u(x-3)]$$

$$= 2\left(\frac{1}{s^2}\right) - e^{-3s}\,\mathcal{L}[\underline{g(x)}]$$

$$2\underline{x} = g(x-3)$$

$$2(x+3) = g((x+3)-3)$$

$$2x+6 = g(x)$$

$$\mathcal{L}[g(x)] = \frac{2}{s^2} + \frac{6}{s}$$

$$\mathcal{L}[f(x)] = \frac{2}{s^2} - e^{-3s}\left(\frac{2}{s^2} + \frac{6}{s}\right)$$

$$F(s) =$$

b. $f(x) = \begin{cases} 1 & 0 \le x < 2 \\ x-2 & 2 \le x < 4 \\ e^{-(x-4)} & x \ge 4 \end{cases} = \begin{cases} 1 & 0 \le x < 2 \\ x-2 & 2 \le x < 4 \\ e^{-(x-4)} & x \ge 4 \end{cases}$

$$f(x) = f_1(x) + (f_2(x) - f_1(x))\,u(x-c)$$

$$f(x) = f_1(x) + (f_2(x) - f_1(x))\,u(x-c_1) + (f_3(x) - f_2(x))\,u(x-c_2)$$

$$f(x) = f_1 + \sum_{k=2}^{n} (f_k - f_{k-1})\,u(x - c_{k-1})$$

$$f(x) = 1 + (x-2-1)\,u(x-2) + (e^{-(x-4)} - (x-2))\,u(x-4)$$

$$= 1 + (x-3)\,u(x-2) + (e^{-(x-4)} - x + 2)\,u(x-4)$$

$$F(s) = \frac{1}{s} + \mathcal{L}\left[(x-3)u(x-2)\right] + \mathcal{L}\left[\left(e^{-(x-4)} - x + 2\right)u(x-4)\right]$$

$$g(x-2) = x-3$$

$$g((x+2)-2) = (x+2)-3$$

$$g(x) = x-1$$

$$\mathcal{L}[g(x)] = \frac{1}{s^2} - \frac{1}{s}$$

$$h(x-4) = e^{-(x-4)} - x + 2$$

$$h((x+4)-4) = e^{-((x+4)-4)} - (x+4) + 2$$

$$h(x) = e^{-x} - x - 2$$

$$\mathcal{L}[h(x)] = \frac{1}{s+1} - \frac{1}{s^2} - \frac{2}{s}$$

$$F(s) = \frac{1}{s} + \mathcal{L}\left[g(x-2)u(x-2)\right] + \mathcal{L}\left[h(x-4)u(x-4)\right]$$

$$= \frac{1}{s} + e^{-2s} \mathcal{L}[g(x)] + e^{-4s} \cdot \mathcal{L}[h(x)]$$

$$\boxed{F(s) = \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right) + e^{-4s}\left(\frac{1}{s+1} - \frac{1}{s^2} - \frac{2}{s}\right)}$$

2. Given $F(s)$, find $f(x)$:

a. $F(s)=\dfrac{1+e^{-\pi x}}{s^2+1}$ $= \dfrac{1}{s^2+1} + e^{-\pi x}\left(\dfrac{1}{s^2+1}\right)$

$$e^{-cx}\mathcal{L}[g(x)]$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] + \mathcal{L}^{-1}\left[e^{-\pi x}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \sin x + g(x-c)\,u(x-c)$$

$$c=\pi$$

$$\mathcal{L}[g(x)] = \frac{1}{s^2+1}$$

$$g(x) = \sin x$$

$$g(x-c) = \sin(x-c)$$

$$f(x) = \sin x + \sin(x-\pi)\,u(x-\pi)$$

Piecewise :

$0 \le x < \pi$: $u(x-\pi) = 0$

$$f(x) = \sin x$$

$x \ge \pi$: $u(x-\pi) = 1$

$$f(x) = \sin x + \sin(x-\pi)$$
$$= \sin x - \sin x$$
$$= 0$$

$$\boxed{f(x) = \begin{cases} \sin x & 0 \le x < \pi \\ 0 & x \ge \pi \end{cases}}$$

b. $F(s) = \dfrac{e^{-2s}}{s(s+1)} = e^{-2s}\left[\dfrac{1}{s(s+1)}\right] \overset{PFD}{=} e^{-2s}\left[\dfrac{1}{s} - \dfrac{1}{s+1}\right]$

$$e^{-cs}\,\mathcal{L}[g(x)]$$

$$c = 2$$

$$\mathcal{L}[g(x)] = \frac{1}{s} - \frac{1}{s+1}$$

$$g(x) = 1 - e^{-x}$$

$$g(x-2) = 1 - e^{-(x-2)}$$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right)\right]$$

$$f(x) = g(x-2)\,u(x-2)$$

$$= \left(1 - e^{-(x-2)}\right)u(x-2)$$

Piecewise: if $0 \le x < 2$: $u(x-2) = 0$

$f(x) = 0$

if $x \geq 2$: $u(x-2) = 1$

$f(x) = 1 - e^{-(x-2)}$

$$\boxed{f(x) = \begin{cases} 0 & 0 \le x < 2 \\ 1 - e^{-(x-2)} & x \geq 2 \end{cases}}$$