Math 3321 – Lecture 14 notes

Table of Laplace Transforms

$f(x) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(x)]$
1	$\frac{1}{s}, \quad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

And we discovered last time that:

$$\mathcal{L}[f(x)] = \int_0^\infty e^{-sx} f(x)\,dx$$

$$\mathcal{L}[y'(x)] = -y(0) + s\mathcal{L}[y(x)]$$

$$\mathcal{L}[y''(x)] = -y'(0) - sy(0) + s^2\mathcal{L}[y(x)]$$

Some properties:

$$\mathcal{L}[\alpha f(x)] = \alpha F(s)$$

$$\mathcal{L}[f(x) \pm g(x)] = F(s) \pm G(s)$$

Note: $\mathcal{L}[f(x)g(x)]$ is generally not equal to $\mathcal{L}[f(x)]\mathcal{L}[g(x)]$

Example:

$$\frac{1}{s} = \mathcal{L}[1] = \mathcal{L}[1 \cdot 1]$$

$$\neq \mathcal{L}[1] \cdot \mathcal{L}[1]$$

$$= \frac{1}{s} \cdot \frac{1}{s}$$

$$= \frac{1}{s^2}$$

The Laplace Transform is typically used to solve problems of the form:
$y'' + ay' + by = f(x), y(0) = \alpha, y'(0) = \beta$ where $f(x)$ is a piecewise defined function.

Using the information above, we could solve for the Laplace transform of $y(x)$ given $y'' + ay' + by = f(x), y(0) = \alpha, y'(0) = \beta$ (an initial value problem). So, the question is, what is $y(x)$? This is where *Inverse Laplace Transforms* comes in.

DEFINITION If *F(s)* is a given transform and if the function *f*, continuous on $[0, \infty)$, has the property that $\mathcal{L}[f(x)] = F(s)$, then *f* is called the **inverse Laplace transform** of *F(s)*, and is denoted by $f(x) = \mathcal{L}^{-1}[F(s)]$

The operator $\mathcal{L}^{-1}$ is called the inverse operator of $\mathcal{L}$

We will use the table on these next examples and "go backwards":

$$e^{\alpha x} \to \frac{1}{s-\alpha}$$

Examples:

Find $\mathcal{L}^{-1}[F(s)]$:

1. $F(s) = \dfrac{1}{s-3}$: $\dfrac{1}{s-\alpha}$, $\alpha = 3$

$$\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = \boxed{e^{3x}}$$

$1 \to \frac{1}{s}$

2. $F(s)=\frac{4}{s}-\frac{3}{s-4}$

$$f(x)=\mathcal{L}^{-1}\left[\frac{4}{s}-\frac{3}{s-4}\right]$$

$$=4\mathcal{L}^{-1}\left[\frac{1}{s}\right]-3\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]$$

$$=4(1)-3(e^{4x})$$

$$=\boxed{4-3e^{4x}}$$

xe^{-2x}

3. $F(s)=\dfrac{2}{(s+2)^2}-\dfrac{s}{s^2-2s+2}$

$x^n e^{rx}, \quad n=1,2,\ldots \qquad \dfrac{n!}{(s-r)^{n+1}}, \qquad s>r$

$$\frac{2}{(s+2)^2}$$

$$\frac{2}{s+2^2}:$$

$$xe^{-2x}$$

$$2\cdot\left[\frac{1}{(s+2)^2}\right]$$

$r=-2,\ n=1$

$$\frac{s}{s^2-2s+2} : \frac{s}{(s^2-2s+1)+1} = \frac{s}{(s-1)^2+1}$$

$e^{\alpha x}\cos\beta x \qquad \dfrac{s-\alpha}{(s-\alpha)^2+\beta^2},$

$\alpha=1,\ \beta=1$

$$\frac{s}{(s-1)^2+1} = \frac{s-1 \quad +1}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

$e^x \cos x \qquad e^x \sin x$

$e^{\alpha x} \sin \beta x \qquad \frac{\beta}{(s-\alpha)^2+\beta^2},$

$$\mathcal{L}^{-1}\left[\frac{2}{(s+2)^2} - \frac{s}{s^2-2s+2}\right] = \mathcal{L}^{-1}\left[2\left(\frac{1}{(s+2)^2}\right) - \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] - \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= \boxed{2xe^{-2x} - e^x \cos x - e^x \sin x}$$

Need Partial Fraction Decomposition!

4. $F(s) = \frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1)+(Bs+C)(s+1)}{(s+1)(s^2+1)}$

$$1 = A(s^2+1) + Bs(s+1) + C(s+1)$$

$s=-1: \; 1 = 2A \rightarrow A = \frac{1}{2}$

$s=0: \; 1 = A + C \rightarrow C = \frac{1}{2}$

s^2 terms: $0 = A + B \rightarrow B = -\frac{1}{2}$

$$F(s) = \frac{(\frac{1}{2})}{s+1} - \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{2}}{s^2+1}$$

$$f(x) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{2}\mathcal{L}\left[\frac{1}{s^2+1}\right]$$

$$= \boxed{\frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x}$$

5. $F(s)=\dfrac{s^2-3s-1}{s^3+s^2-2s}$ $=\dfrac{s^2-3s-1}{s(s^2+s-2)}$ $=\dfrac{s^2-3s-1}{s(s+2)(s-1)}$

$$\frac{s^2-3s-1}{s^3+s^2-2s}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-1}$$

$$s^2-3s-1=A(s+2)(s-1)+Bs(s-1)+Cs(s+2)$$

$s=-2$: $9=6B \rightarrow B=\frac{3}{2}$

$s=1$: $-3=3C \rightarrow C=-1$

$s=0$: $-1=-2A \rightarrow A=\frac{1}{2}$

$$F(s)=\frac{\left(\frac{1}{2}\right)}{s}+\frac{\left(3/2\right)}{s+2}-\frac{1}{s-1}$$

$$\boxed{f(x)=\frac{1}{2}+\frac{3}{2}e^{-2x}-e^{x}}$$

$Y = \mathcal{L}[y]$

$\mathcal{L}[y'] = -y(0) + s\,\mathcal{L}[y]$

$= -y(0) + sY$

$\mathcal{L}[y''] = -y'(0) - s\,y(0) + s^2 Y$

Find the solution of the initial value problem:

6. $y'' - y = e^x$, $y(0) = 1$, $y'(0) = 0$

$\mathcal{L}[y''] - \mathcal{L}[y] = \mathcal{L}[e^x]$

$(-y'(0) - s\,y(0) + s^2 Y) - Y = \frac{1}{s-1}$

$\quad\;\; 0 \qquad\quad\;\; 1$

$-s + s^2 Y - Y = \frac{1}{s-1}$

$s^2 Y - Y = \frac{1}{s-1} + s$

$Y(s^2 - 1) = \frac{1}{s-1} + s$

$Y = \frac{1}{(s^2-1)(s-1)} + \frac{s}{(s^2-1)}$

$$Y = \frac{1}{(s^2-1)(s-1)} + \frac{s}{s^2-1} = \frac{1 + s(s-1)}{(s^2-1)(s-1)}$$

$$= \frac{s^2-s+1}{(s^2-1)(s-1)}$$

PFD: $\dfrac{s^2-s+1}{(s^2-1)(s-1)} = \dfrac{s^2-s+1}{(s+1)(s-1)(s-1)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+1}$

$$s^2-s+1 = A(s-1)(s+1) + B(s+1) + C(s-1)^2$$

$s=1$: $1 = 2B \rightarrow B = 1/2$

$s=-1$: $3 = 4C \rightarrow C = 3/4$

s^2 terms: $1 = A + C \rightarrow A = 1/4$

$$Y = \frac{1}{4}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{(s-1)^2}\right) + \frac{3}{4}\left(\frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}[Y] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$x^n e^{rx}, \quad n = 1, 2, \ldots \qquad \frac{n!}{(s-r)^{n+1}},$ ← $n=1, r=1$

$$y = \frac{1}{4}e^{x} + \frac{1}{2}xe^{x} + \frac{3}{4}e^{-x}$$

Pop 08: E E E E E