Math 3321 – Lecture 13 notes

Sections 4.1 & 4.2 Intro to Laplace Transforms

Laplace transforms can be used to turn linear constant coefficient differential equations into algebraic equations.

Definition: Let *f* be a continuous function on the interval $[0, \infty)$. The Laplace transform of *f*, denoted by $\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)\,dx$. The domain of *F* is the set of all real numbers *s* for which the improper integral converges.

Examples:

$$\mathcal{L}\left[e^x\right] = \int_0^\infty e^{-sx} \cdot e^x \, dx$$

$$= \int_0^\infty e^{x-sx} \, dx$$

$$= \frac{1}{1-s}\int_0^\infty (1-s) e^{(1-s)x} \, dx$$

$u = (1-s)x$

$du = (1-s)dx$

$$= \frac{1}{1-s}\left[e^{(1-s)x}\right]_0^\infty$$

only converges if $s > 1$

$\rightarrow (1-s) < 0$

$$= \frac{1}{1-s} \lim_{x \to \infty} e^{(1-s)x} - \frac{1}{1-s} e^0$$

$$= 0 - \frac{1}{1-s} = \boxed{\frac{1}{s-1}}$$

$$\mathcal{L}\left[e^{\alpha x}\right] = \int_0^\infty e^{-sx} e^{\alpha x}\, dx$$

$$= \int_0^\infty e^{(\alpha - s)x}\, dx$$

$$= \frac{1}{\alpha - s} e^{(\alpha - s)x} \Big|_0^\infty$$

$$= \lim_{b \to \infty} \frac{1}{\alpha - s} \left[e^{(\alpha - s)b} - e^0 \right]$$

need $\alpha - s < 0$

$\alpha < s$

$$= \frac{1}{\alpha - s}\left[0 - 1\right] = \boxed{\frac{1}{s - \alpha}}$$

$$\mathcal{L}[1] = \int_0^\infty e^{-sx} \cdot 1 \, dx$$

$$= \int_0^\infty e^{-sx} \, dx$$

$$= -\frac{1}{s} e^{-sx} \Big|_0^\infty$$

$s > 0$

$$= \lim_{b \to \infty} -\frac{1}{s}\left[e^{-sb} - e^0\right]$$

$$= -\frac{1}{s}[0 - 1]$$

$$= \frac{1}{s}$$

$1 = e^{0x}$

$\mathcal{L}[1] = \mathcal{L}[e^{0x}]$

$= \frac{1}{s - 0}$

$= 1/s$

The Laplace transform is a **linear** transformation.

$$\mathcal{L}\left[\beta e^{\alpha x}\right] = \beta \, \mathcal{L}\left[e^{\alpha x}\right]$$

$$= \beta \left(\frac{1}{s-\alpha} \right) \qquad \alpha < s$$

$$= \frac{\beta}{s-\alpha}$$

$$\mathcal{L}[y'(x)] = \int_0^\infty e^{-sx}\, y'(x)\,dx$$

IBP

$$u = e^{-sx} \qquad v = y(x)$$

$$du = -se^{-sx} \qquad dv = y'(x)\,dx$$

$$= \lim_{b\to\infty} \int_0^b e^{-sx}\, y'(x)\,dx$$

$$= \lim_{b\to\infty} \left[y e^{-sx} \Big|_0^b - \int_0^b -sye^{-sx}\,dx \right]$$

$$= \lim_{b\to\infty} \left[y(b)e^{-sb} - y(0)e^{0} \right] + s \underbrace{\int_0^\infty e^{-sx}\, y(x)\,dx}$$

$$= [0 - y(0)] + s\,\mathcal{L}[y(x)]$$

$$= \boxed{-y(0) + s\,\mathcal{L}[y(x)]}$$

$$\mathcal{L}[y'] = -y_0 + s\mathcal{L}[y]$$

$$\mathcal{L}[y''(x)] = \mathcal{L}[(y'(x))']$$

$$= -y'(0) + s\mathcal{L}[y'(x)]$$

$$= -y'(0) + s[-y(0) + s\mathcal{L}[y]]$$

$$= \boxed{-y'(0) - sy(0) + s^2\mathcal{L}[y]}$$

$$L\left[y'(x)\right] = -y(0) + sL\left[y(x)\right]$$

$$L\left[y''(x)\right] = -y'(0) - s\,y(0) + s^2 L\left[y(x)\right]$$

$$L\left[f(x)\right] = \int_0^{\infty} e^{-sx} f(x)\,dx$$

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
1	$\frac{1}{s}, \quad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

$$\mathcal{L}[\sin(\beta x)] = \frac{\beta}{s^2+\beta^2} \rightarrow \mathcal{L}[\sin x] = \frac{1}{s^2+1}$$

Using the table, find

$$\mathcal{L}\left[2\sin x - 3e^{-x} + 1\right] = 2\,\mathcal{L}[\sin x] - 3\,\mathcal{L}[e^{-x}] + \mathcal{L}[1]$$

$$= 2\left(\frac{1}{s^2+1}\right) - 3\left(\frac{1}{s+1}\right) + \frac{1}{s}$$

$$= \boxed{\frac{2}{s^2+1} - \frac{3}{s+1} + \frac{1}{s}}$$

$$\mathcal{L}\left[3e^{2x}\cos(3x)+x\sin(x)-3x^2\right]= 3\mathcal{L}\left[e^{2x}\cos(3x)\right]+\mathcal{L}\left[x\sin x\right]$$

$$-3\mathcal{L}\left[x^2\right]$$

$\alpha=2$, $\beta=3$

$e^{\alpha x}\cos\beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$,	$s>\alpha$

$$\mathcal{L}\left[e^{2x}\cos(3x)\right]=\frac{s-2}{(s-2)^2+3^2}=\frac{s-2}{(s-2)^2+9}$$

$x\sin\beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}$,	$s>0$

$$\mathcal{L}\left[x\sin x\right]=\frac{2s}{(s^2+1)^2}$$

$\beta=1$

x^n, $n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$,	$s>0$

$$\mathcal{L}\left[x^2\right]=\frac{2!}{s^3}=\frac{2}{s^3}$$

$n=2$

$$3\left(\frac{s-2}{(s-2)^2+9}\right)+\frac{2s}{(s^2+1)^2}-3\left(\frac{2}{s^3}\right)$$

$$=\boxed{\frac{3s-6}{(s-2)^2+9}+\frac{2s}{(s^2+1)^2}-\frac{6}{s^3}}$$

Example 2. Find the Laplace transform $\mathcal{L}[y(x)] = Y(s)$ of the solution of the initial-value problem

$$y' - 2y = 2e^{-3x}; \quad y(0) = -2.$$

$$\mathcal{L}[y'] - 2\mathcal{L}[y] = 2\mathcal{L}[e^{-3x}]$$

$$-y(0) + s\mathcal{L}[y] - 2\mathcal{L}[y] = 2\left(\frac{1}{s+3}\right)$$

$$2 + (s-2)\mathcal{L}[y] = \frac{2}{s+3}$$

$$(s-2)\mathcal{L}[y] = \frac{2}{s+3} - 2$$

$$\boxed{\mathcal{L}[y] = \frac{2}{(s+3)(s-2)} - \frac{2}{s-2}}$$