Math 3321 – Lecture 12 notes

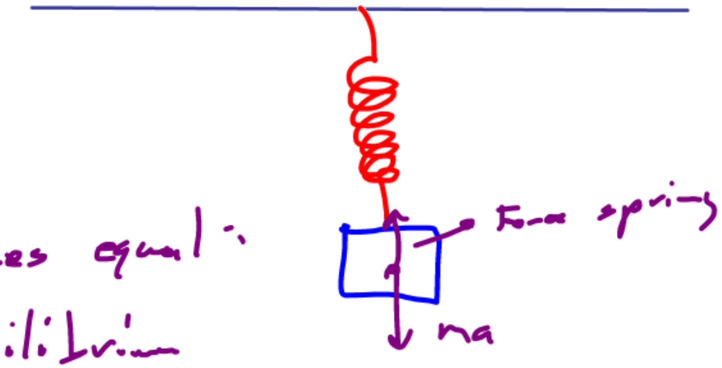
no resistance

no external forces

Section 3.6. Vibrating Mechanical Systems

I. Undamped Free Vibrations

Forces equal: equilibrium



Hooke's Law: The restoring force of a spring is proportional to the displacement: from equilibrium point

$F = -ky,\ k > 0$

y: displacement

Newton's Second Law: Force equals mass times acceleration:

$$F = ma = m\frac{d^2y}{dt^2}$$

Combining the two: $\boxed{m\dfrac{d^2y}{dt^2} = -ky}$ $\rightarrow$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

This can be re-written as: $\dfrac{d^2y}{dt^2} + \omega^2 y = 0$ where $\omega = \sqrt{k/m}$

$$\rightarrow y'' + \omega^2 y = 0$$

$$\lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

$\dfrac{\omega}{2\pi}$ is called the natural frequency of the system.

Solving this for a general solution:

$$y'' + \omega^2 y = 0$$

$$\lambda^2 + \omega^2 = 0$$

$$\lambda = \pm i\omega$$

$$y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\text{period} = \frac{2\pi}{\omega} = T$$

$$\text{Frequency} = \frac{1}{T}$$

This general solution can be re-written as

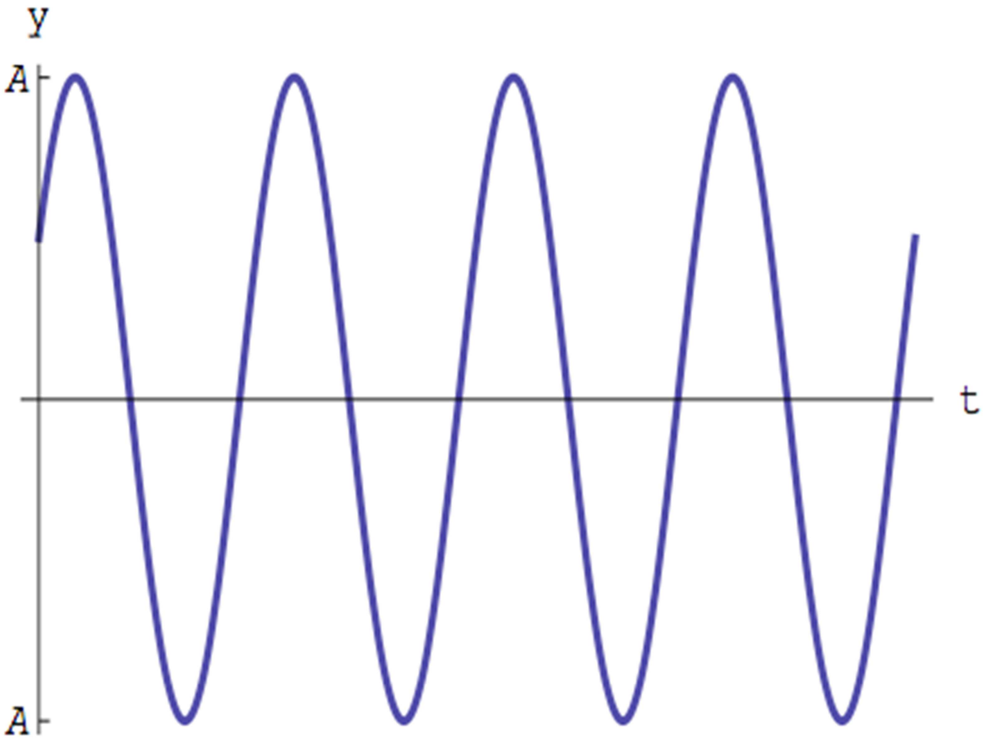
$$y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\boxed{y = A \sin(\omega t + \phi_0)}$$

How? $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\alpha = \phi_0$, $\beta = \omega t$

So the graph looks like this:



$$y = A\sin(\omega t + \phi_0)$$

A = amplitude

period $= \dfrac{2\pi}{\omega}$

freq $= \dfrac{\omega}{2\pi}$

ϕ_0 = phase shift

friction, air resistance

II. Damped, Free Vibrations:

$v = y'$

A resistance force R proportional to the velocity ~~$v=y'$~~ and acting in a direction opposite to the motion: $R = -cy'$, $c > 0$

Force Equation: $F = -ky - cy'$

Newton's Second Law: $F = ma = my''$

Combine to get the mathematical model: $my'' = -ky - cy'$ or $\boxed{y'' + \frac{c}{m}y' + \frac{k}{m}y = 0}$

c, k, m are constants.

Lets look at the characteristic polynomial and its roots:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \rightarrow \lambda = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - \frac{4k}{m}}}{2}$$

key: $D = c^2 - 4km$

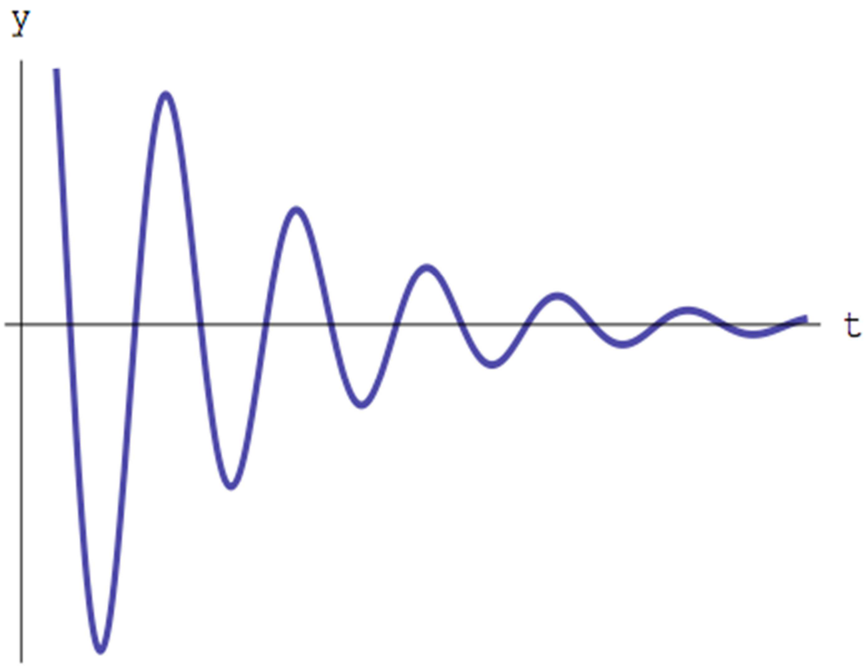
$$\lambda = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$$

We will have 3 cases based on the nature of the roots.

Case 1: Complex roots $c^2 - 4km < 0$

This case is called **Underdamped**

$$\lambda = -\frac{c}{2m} \pm i\,\frac{\sqrt{4km - c^2}}{2m}$$



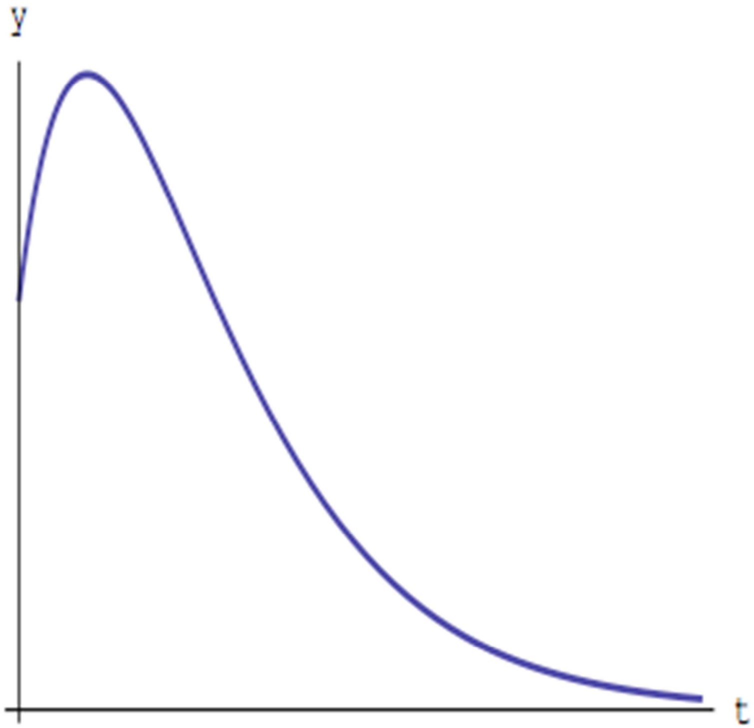
$$y_1 = e^{-\frac{c}{2m}t} \cos \frac{\sqrt{4km-c^2}}{2m} t$$

$$y_2 = e^{-\frac{c}{2m}t} \sin \frac{\sqrt{4km-c^2}}{2m} t$$

$$\boxed{y = C_1 \underline{e^{-\frac{c}{2m}t}} \cos \frac{\sqrt{4km-c^2}}{2m} t + C_2 e^{-\frac{c}{2m}t} \sin \frac{\sqrt{4km-c^2}}{2m} t}$$

$e^{-\frac{c}{2m}t} \xrightarrow{t \to \infty} 0$, Amplitude $\xrightarrow[t \to \infty]{} 0$

Case 2: Two distinct real roots
This case is called **Overdamped**



$$D = c^2 - 4km > 0$$

$$\lambda = -\frac{c}{2m} \pm \frac{\sqrt{c^2-4km}}{2m}$$

$$\left.\begin{aligned} \lambda_1 &= -\frac{c}{2m} + \frac{\sqrt{c^2-4km}}{2m} \\ \lambda_2 &= -\frac{c}{2m} - \frac{\sqrt{c^2-4km}}{2m} \end{aligned}\right] \to < 0$$

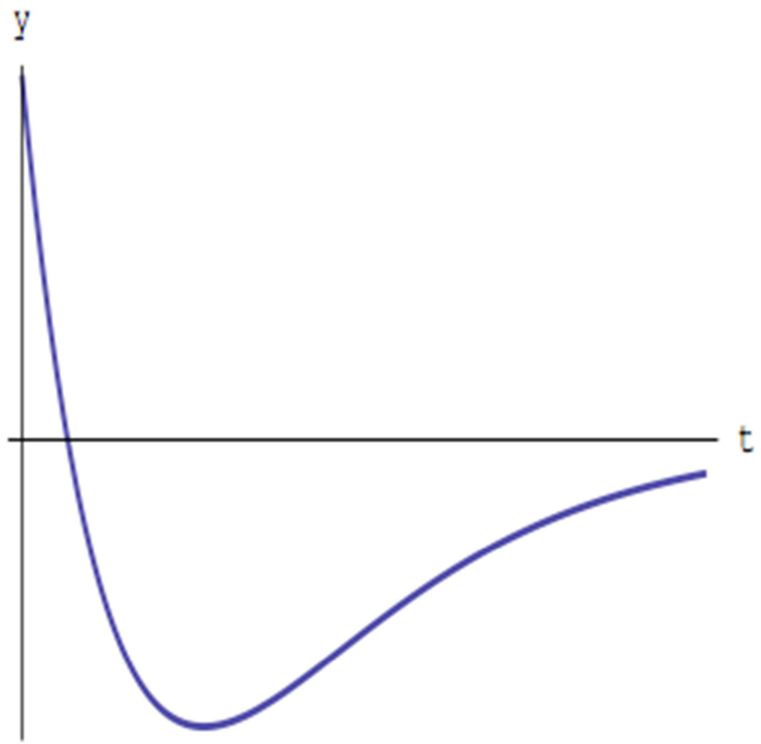
$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$y = C_1 e^{-|\lambda_1| t} + C_2 e^{-|\lambda_2| t}$$

no sin/cos $\to$ no oscillations

$e^{-|\lambda_1| t} \xrightarrow{t \to \infty} 0$, decays to equilibrium

Case 3: One real root
This case is called **Critically Damped**



$$D = c^2 - 4km = 0$$

$$\lambda = -\frac{c}{2m} \quad (\text{repeated})$$

$$y = C_1 e^{-\frac{c}{2m}t} + C_2 t e^{-\frac{c}{2m}t}$$

decay to equilibrium
(very rapid)

1–3) c, c, c

Pop 07: 4) A
5) A

III. Forced Free Vibrations

Apply an external force G to an undamped, freely vibrating system.

Force Equation: $F = -ky + G$

$\Rightarrow my'' = -ky + G$

re-write as a nonhomogeneous equation: $y'' + \frac{k}{m} y = \frac{G}{m}$

A periodic external force: $G = F_0 \cos(\gamma t)$, $F_0, \gamma > 0 \; (const)$

This makes the force equation above into: $F = -ky + F_0 \cos(\gamma t)$

$$\Rightarrow y'' + \frac{k}{m} y = \frac{F_0}{m} \cos(\gamma t)$$

$$\underbrace{y'' + \omega^2 y}_{} = \frac{F_0}{m} \cos(\gamma t)$$

remember ω ? $= \sqrt{\frac{k}{m}}$

$y_1 = \cos(\omega t)$, $y_2 = \sin(\omega t)$

$\frac{\omega}{2\pi}$ is called the natural frequency of the system and $\frac{\gamma}{2\pi}$ is called the applied frequency of the system.

2 cases: $\gamma \neq \omega$ or $\gamma = \omega$

Case 1: $\gamma \neq \omega$

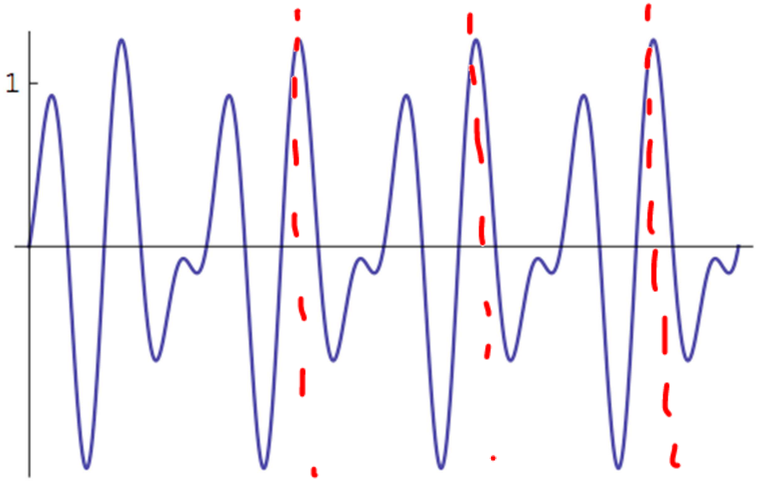
$$y'' + \omega^2 y = \frac{F_0}{m}\cos(\gamma t)$$

General solution of the reduced equation: $y = A\sin(\omega t + \phi_0)$ and the *form* of the particular solution (found by undetermined coefficients) $z = A\cos(\gamma t) + B\sin(\gamma t)$

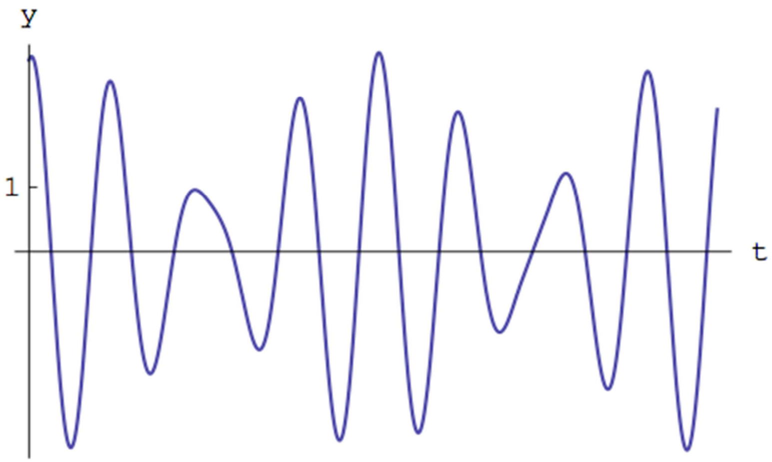
A particular solution is $z = \frac{F_0 / m}{\omega^2 - \gamma^2}\cos(\gamma t)$

So the general solution is $y = A\sin(\omega t + \phi_0) + \frac{F_0 / m}{\omega^2 - \gamma^2}\cos(\gamma t)$

ω/γ rational: periodic motion



ω/γ irrational: not periodic



Case 2: $\gamma = \omega$

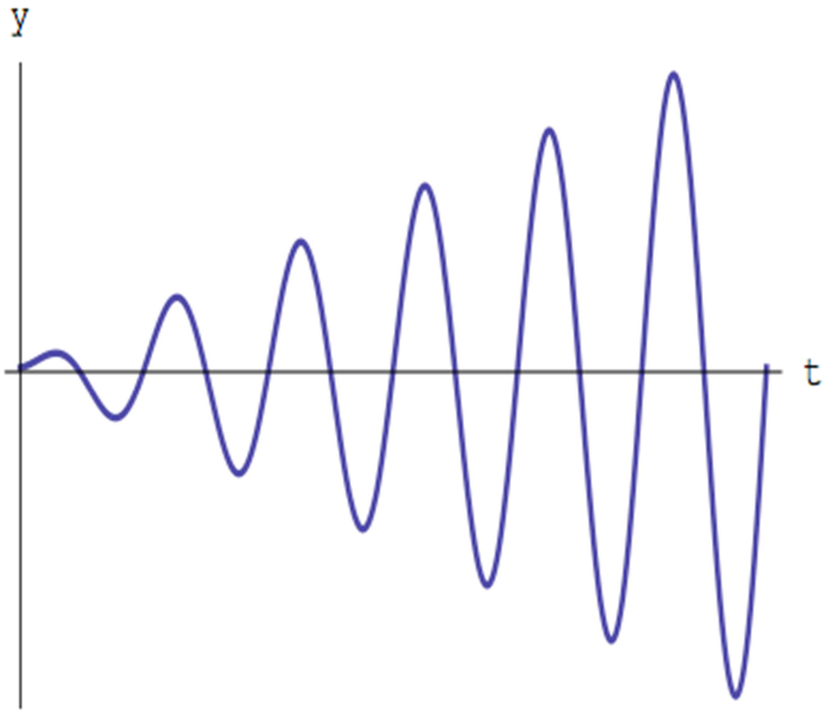
$$\boxed{y'' + \omega^2 y = \frac{F_0}{m}\cos(\omega t)}$$

General solution of the reduced equation: $y = A\sin(\omega t + \phi_0)$ and the *form* of the particular solution (found by undetermined coefficients) $z = A\cos(\omega t) + B\sin(\omega t)$

A particular solution is $z = \dfrac{F_0}{2\omega m} t\sin(\omega t)$

So the general solution is $y = A\sin(\omega t + \phi_0) + \dfrac{F_0}{2\omega m} t\sin(\omega t)$

Resonance: Unbounded oscillation!



[Youtube: Tacoma Narrows Bridge Collapse]

IV. Forced Damped Vibrations

Apply an external force G to a damped, freely vibrating system

Force Equation: $F = -ky - cy' + G$

$\Rightarrow my'' = -ky - cy' + G$

$$\boxed{\Rightarrow y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{G}{m}}$$

A periodic external force: $G = F_0 \cos(\gamma t)$, $F_0, \gamma > 0$ $(const)$

Makes the Force Equation: $F = -ky - cy' + F_0 \cos(\gamma t)$

$$F = -ky - cy' + F_0 \cos(\gamma t)$$

$$\boxed{\Rightarrow y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_0}{m}\cos(\gamma t)}$$

$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

General solution, reduced equation:

Case I: two distinct real roots: $Y_c(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, $r_1 < 0, r_2 < 0$

Note: $\lim_{t\to\infty} Y_c(t) = 0$ exponentially

Case 2: one repeated real root: $Y_c(t) = C_1 e^{rt} + C_2 t e^{rt}$, $r < 0$

Note: $\lim_{t\to\infty} Y_c(t) = 0$ exponentially

Case 3: complex roots: $Y_c(t) = C_1 e^{rt}\cos(\omega t) + C_2 e^{rt}\sin(\omega t)$, $r_1 = r + \omega i, r_2 = r - \omega i, r < 0$

Note: $\lim_{t\to\infty} Y_c(t) = 0$ oscillations decay exponentially

Particular solution of $y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{F_0}{m} \cos(\gamma t)$ (N) will have the form

$Z(t) = A\cos(\gamma t) + B\sin(\gamma t)$

So the general solution will look like $y(t) = Y_c(t) + Z(t)$

Note: $\lim_{t\to\infty} y(t) = Z(t)$

$Y_c(t)$, the general solution of the reduced equation, is called the **transient solution**.

$Z(t)$ the particular solution of (N), is called the **steady state solution**.

$$y = A\sin(\omega t + \phi_0)$$

Examples:

An object is in simple harmonic motion. Find an equation for the motion given that the period is $2\pi/3$ and at time $t = 0$, $y = 1$, and $y' = 3$. What is the equation of motion?

$T = \frac{2\pi}{3}$ $\quad y(0) = 1$ $\quad y'(0) = 3$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{3}$$

$$\boxed{\omega = 3}$$

$$y = A\sin(3t + \phi_0)$$

$$y(0) = A\sin\phi_0 = 1 \rightarrow A = \frac{1}{\sin\phi_0}$$

$$y' = 3A\cos(3t + \phi_0)$$

$$y'(0) = 3A\cos\phi_0 = 3 \rightarrow A\cos\phi_0 = 1$$

$$\frac{\cos\phi_0}{\sin\phi_0} = 1$$

$$\cot\phi_0 = 1 \rightarrow \tan\phi_0 = 1$$

$$\boxed{\phi_0 = \frac{\pi}{4}}$$

$$A = \frac{1}{\sin\phi_0} = \frac{1}{\sin\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \boxed{\sqrt{2} = A}$$

$$\boxed{y = \sqrt{2}\sin\left(3t + \frac{\pi}{4}\right)}$$

$$y = A\sin(\omega t + \phi_0)$$

An object is in simple harmonic motion. Find an equation for the motion given that the frequency is $5/\pi$ and at time $t = 0$, $y = 1$, and $y' = 0$. What is the equation of motion?

$f = 5/\pi$ $\qquad$ $y(0) = 1$, $y'(0) = 0$

$$\text{period} = \frac{1}{f} = \frac{\pi}{5}$$

$$\frac{2\pi}{\omega} = \frac{\pi}{5}$$

$$10\pi = \omega\pi$$

$$10 = \omega$$

$$y = A\sin(10t + \phi_0)$$

$$y(0) = A\sin\phi_0 = 1$$

$$A = \frac{1}{\sin\phi_0}$$

$$y' = 10A\cos(10t + \phi_0)$$

$$y'(0) = 10A\cos(\phi_0) = 0$$

$$A\cos(\phi_0) = 0$$

$$\cot\phi_0 = 0$$

$$\cot \phi_0 = 0 \rightarrow \frac{\cos \phi_0}{\sin \phi_0} = 0 \rightarrow \cos \phi_0 = 0$$

$$\boxed{\phi_0 = \frac{\pi}{2}}$$

$$A = \frac{1}{\sin \phi_0} = \frac{1}{1} = \boxed{1 = A}$$

$$\boxed{y = \sin\left(10t + \frac{\pi}{2}\right)}$$