

Section 3.7: Higher Order Linear ODE's

An n -th order linear differential equation is in the form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

If $f(x) = 0$, then it is *homogeneous* and if $f(x) \neq 0$, then it is *nonhomogeneous*.

Everything we stated for second order linear ODE applies to n -th order linear ODE.

To find the Wronskian of our n solutions:

$$W[y_1, y_2, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \dots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} \end{vmatrix}$$

Given

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_1y' + a_0y = 0$$

where a_0, a_1, \dots, a_{n-1} are real numbers.

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

is called the *characteristic equation* of (3); the n^{th} -degree polynomial $p(\lambda)$ is called the *characteristic polynomial*. The roots of the characteristic equation are called the *characteristic roots*.

10. Find the general solution of $y''' + 3y'' - 6y' - 8y = 0$ if $\lambda = 2$ is a root of the characteristic polynomial.

$$\lambda^3 + 3\lambda^2 - 6\lambda - 8 = 0$$

one root is $\lambda = 2$

How to find the rest?

polynomial long division

synthetic division : $\underline{2}$

1	3	-6	-8	
	2	10	8	
1	5	4	0	0
$\lambda^2 + 5\lambda + 4$				

$$\begin{aligned}\lambda^3 + 3\lambda^2 - 6\lambda - 8 &= (\lambda - 2)(\lambda^2 + 5\lambda + 4) \\ &= (\lambda - 2)(\lambda + 1)(\lambda + 4)\end{aligned}$$

roots : $\lambda = 2, -1, -4$

$$y_1 = e^{2x}, \quad y_2 = e^{-x}, \quad y_3 = e^{-4x}$$

$$y = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{-4x}$$

Pop 07: 1) C
2) C
3) C

11. Find the general solution of $y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$ if $\lambda = -1 + 3i$ is a root of the characteristic polynomial.

$$\lambda^4 + 2\lambda^3 + 9\lambda^2 - 2\lambda - 10 = 0$$

$$\lambda_1 = -1 + 3i \rightarrow \lambda_2 = -1 - 3i$$

$$(\lambda - (-1 + 3i)) (\lambda - (-1 - 3i))$$

$$\lambda^2 + 2\lambda + 10$$

polynomial long division:

$$\begin{array}{r} \lambda^2 - 1 \\ \lambda^2 + 2\lambda + 10 \overline{) \lambda^4 + 2\lambda^3 + 9\lambda^2 - 2\lambda - 10} \\ \underline{-(\lambda^4 + 2\lambda^3 + 10\lambda^2)} \\ -\lambda^2 - 2\lambda - 10 \\ \underline{-(-\lambda^2 - 2\lambda - 10)} \\ 0 \end{array}$$

$$\lambda^4 + 2\lambda^3 + 9\lambda^2 - 2\lambda - 10 = (\lambda^2 + 2\lambda + 10)(\lambda^2 - 1)$$

$$= (\lambda^2 + 2\lambda + 10)(\lambda - 1)(\lambda + 1)$$

$$\hookrightarrow \underline{\lambda = -1 \pm 3i}$$

$$\hookrightarrow \underline{\lambda = 1}$$

$$\hookrightarrow \underline{\lambda = -1}$$

$$y = C_1 e^{-x} \cos 3x + C_2 e^{-x} \sin 3x + C_3 e^x + C_4 e^{-x}$$

12. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos(2x) + C_4 \sin(2x)$ is the general solution of what homogeneous equation?

$$C_1 e^{2x} \rightarrow \lambda = 2 \rightarrow (\lambda - 2)$$

$$C_2 e^{-2x} \rightarrow \lambda = -2 \rightarrow (\lambda + 2)$$

$$C_3 \cos(2x) + C_4 \sin(2x) \rightarrow \lambda = \pm 2i \rightarrow (\lambda - 2i), (\lambda + 2i)$$

$$p(\lambda) = (\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i)$$

$$= (\lambda^2 - 4)(\lambda^2 + 4)$$

$$= \lambda^4 - 16 = 0$$

$$y^{(4)} - 16y = 0$$

13. Give the form of a particular solution of $y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5e^{2x} + \sin(2x) + 6$

$$\lambda^4 + 4\lambda^3 + 13\lambda^2 + 36\lambda + 36 = 0$$

$$(\lambda^4 + 13\lambda^2 + 36) + (4\lambda^3 + 36\lambda) = 0$$

$$(\lambda^2 + 4)(\lambda^2 + 9) + 4\lambda(\lambda^2 + 9) = 0$$

$$(\lambda^2 + 4\lambda + 4)(\lambda^2 + 9) = 0$$

$$(\lambda + 2)^2(\lambda^2 + 9) = 0$$

↑

$$\lambda = -2$$

$$y_1 = e^{-2x}$$

$$y_2 = xe^{-2x}$$

↑

$$\lambda = \pm 3i$$

$$y_3 = \cos 3x$$

$$y_4 = \sin 3x$$

$$f(x) = 5e^{2x} + \sin(2x) + 6$$

$$z = Ae^{2x} + B\sin 2x + C\cos 2x + D$$

14. Find the general solution of $y^{(6)} - y'' = 0$

$$\lambda^6 - \lambda^2 = 0$$

$$\lambda^2(\lambda^4 - 1) = 0$$

$$\lambda^2(\lambda^2 - 1)(\lambda^2 + 1) = 0$$

$$\lambda^2(\lambda - 1)(\lambda + 1)(\lambda^2 + 1) = 0$$

roots: 0 twice, 1, -1, i, -i

$e^0 = 1, xe^0 = x$

$$y_1 = 1, y_2 = x, y_3 = e^x, y_4 = e^{-x}, y_5 = \cos x, y_6 = \sin x$$

$$y = C_1 + C_2x + C_3e^x + C_4e^{-x} + C_5\cos x + C_6\sin x$$