Math 3321 – Lecture 11 Notes

Section 3.7: Higher Order Linear ODE's

An n-th order linear differential equation is in the form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

If $f(x)=0$, then it is *homogeneous* and if $f(x)\neq 0$, then it is *nonhomogeneous*.

Everything we stated for second order linear ODE applies to n-th order linear ODE.

To find the Wronskian of our n solutions:

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

Given

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, \; a_1, \; \ldots, \; a^{n-1}$ are real numbers.

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

is called the *characteristic equation* of (3); the n^{th}-degree polynomial $p(\lambda)$ is called the *characteristic polynomial*. The roots of the characteristic equation are called the *characteristic roots*.

10. Find the general solution of $y''' + 3y'' - 6y' - 8y = 0$ if $\lambda = 2$ is a root of the characteristic polynomial.

$$\lambda^3 + 3\lambda^2 - 6\lambda - 8 = 0$$

one root is $\lambda = 2$

How to find the rest?

polynomial long division

synthetic division :

$$\begin{array}{r|rrr|r} 2 & 1 & 3 & -6 & -8 \\ & & 2 & 10 & 8 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

$\lambda^2 + 5\lambda + 4$

$$\lambda^3 + 3\lambda^2 - 6\lambda - 8 = (\lambda - 2)(\lambda^2 + 5\lambda + 4)$$
$$= (\lambda - 2)(\lambda + 1)(\lambda + 4)$$

roots : $\lambda = 2, -1, -4$

$y_1 = e^{2x}$, $y_2 = e^{-x}$, $y_3 = e^{-4x}$

$$\boxed{y = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{-4x}}$$

Pop07: 1) C

2) C

3) C

11. Find the general solution of $y^{(4)}+2y'''+9y''-2y'-10y=0$ if $\lambda = -1+3i$ is a root of the characteristic polynomial.

$$\lambda^4+2\lambda^3+9\lambda^2-2\lambda-10=0$$

$$\lambda_1=-1+3i \longrightarrow \lambda_2=-1-3i$$

$$(\lambda-(-1+3i))(\lambda-(-1-3i))$$

$$\lambda^2+2\lambda+10$$

polynomial long division:

$$\begin{array}{r} \lambda^2-1 \\ \lambda^2+2\lambda+10 \overline{)\lambda^4+2\lambda^3+9\lambda^2-2\lambda-10} \\ -(\lambda^4+2\lambda^3+10\lambda^2) \\ \hline -\lambda^2-2\lambda-10 \\ -(-\lambda^2-2\lambda-10) \\ \hline 0 \end{array}$$

$$\lambda^4+2\lambda^3+9\lambda^2-2\lambda-10=(\lambda^2+2\lambda+10)(\lambda^2-1)$$

$$= (\lambda^2 + 2\lambda + 10)(\lambda - 1)(\lambda + 1)$$

$\hookrightarrow \lambda = -1 \pm 3i$

$\hookrightarrow \lambda = 1$

$\hookrightarrow \lambda = -1$

$$\boxed{y = C_1 e^{-x} \cos 3x + C_2 e^{-x} \sin 3x + C_3 e^{x} + C_4 e^{-x}}$$

12. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos(2x) + C_4 \sin(2x)$ is the general solution of what homogeneous equation?

$C_1 e^{2x} \rightarrow \lambda = 2 \rightarrow (\lambda - 2)$

$C_2 e^{-2x} \rightarrow \lambda = -2 \rightarrow (\lambda + 2)$

$C_3 \cos(2x) + C_4 \sin(2x) \rightarrow \lambda = \pm 2i \rightarrow (\lambda - 2i), (\lambda + 2i)$

$$p(\lambda) = (\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i)$$

$$= (\lambda^2 - 4)(\lambda^2 + 4)$$

$$= \lambda^4 - 16 = 0$$

$$\boxed{y^{(4)} - 16y = 0}$$

13. Give the form of a particular solution of $y^{(4)}+4y'''+13y''+36y'+36y=5e^{2x}+\sin(2x)+6$

$$\lambda^4+4\lambda^3+13\lambda^2+36\lambda+36=0$$

$$(\lambda^4+13\lambda^2+36)+(4\lambda^3+36\lambda)=0$$

$$(\lambda^2+4)(\lambda^2+9)+4\lambda(\lambda^2+9)=0$$

$$(\lambda^2+4\lambda+4)(\lambda^2+9)=0$$

$$(\lambda+2)^2(\lambda^2+9)=0$$

$\uparrow$ $\lambda=-2$

$y_1=e^{-2x}$

$y_2=xe^{-2x}$

$\uparrow$ $\lambda=\pm 3i$

$y_3=\cos 3x$

$y_4=\sin 3x$

$$f(x) = 5e^{2x} + \sin(2x) + 6$$

$$\boxed{z = Ae^{2x} + B\sin 2x + C\cos 2x + D}$$

14. Find the general solution of $y^{(6)} - y'' = 0$

$$\lambda^6 - \lambda^2 = 0$$

$$\lambda^2(\lambda^4 - 1) = 0$$

$$\lambda^2(\lambda^2 - 1)(\lambda^2 + 1) = 0$$

$$\lambda^2(\lambda - 1)(\lambda + 1)(\lambda^2 + 1) = 0$$

roots: $\underline{0 \text{ twice}}$, $\underline{1, -1}$, $\underline{i, -i}$

$$e^0 = 1, \; xe^0 = x$$

$y_1 = 1$, $y_2 = x$, $y_3 = e^x$, $y_4 = e^{-x}$, $y_5 = \cos x$, $y_6 = \sin x$

$$\boxed{y = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} + C_5 \cos x + C_6 \sin x}$$