Math 3321 – Lecture 10 Notes

Section 3.5: Nonhomogeneous ODE's with constant coefficients

$$y'' + ay' + by = f(x) \quad (N)$$
$$y'' + ay' + by = 0 \quad (H)$$

Special Case of

$$y'' + p(x)y' + q(x)y = f(x) \quad (N)$$

$$y'' + p(x)y' + q(x)y = 0 \quad (H)$$

Method of Undetermined Coefficients:

Remark: *Limitations of the method*. In contrast to variation of parameters, which can be applied to any nonhomogeneous equation, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form $y''+ay'+by=f(x)$ where a and b are constants and the nonhomogeneous term f is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions.

A particular solution of $y''+ay'+by=f(x)$

If $f(x)=$	try $z(x)=$
ce^{rx}	Ae^{rx}
$c\cos\beta x+d\sin\beta x$	$z(x)=A\cos\beta x+B\sin\beta x$
$ce^{\alpha x}\cos\beta x+de^{\alpha x}\sin\beta x$	$z(x)=Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x$

Note: The first line includes the case $r=0$;
if $f(x)=ce^{0x}=c$, then $z=Ae^{0x}=A$.

Examples:

1. $y'' - 5y' + 6y = 7e^{-4x}$

(H): $\lambda^2 - 5\lambda + 6 = 0 \rightarrow (\lambda - 2)(\lambda - 3) = 0$

$y_1 = e^{2x}, \quad y_2 = e^{3x}$

guess: $z = Ae^{-4x}$

$z' = -4Ae^{-4x}$

$z'' = 16Ae^{-4x}$

$16Ae^{-4x} + 20Ae^{-4x} + 6Ae^{-4x} = 7e^{-4x}$

$42Ae^{-4x} = 7e^{-4x}$

$A = \frac{1}{6} \rightarrow z = \frac{1}{6}e^{-4x}$

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{6}e^{-4x}$$

2. $y'' - 2y' + y = 3\cos(2x)$

$\lambda^2 - 2\lambda + 1 = 0 \to (\lambda - 1)^2 = 0 \to \{e^x, xe^x\}$

$z = A\cos(2x) + B\sin(2x)$

$z' = -2A\sin(2x) + 2B\cos(2x)$

$z'' = -4A\cos(2x) - 4B\sin(2x)$

$(-4A\cos(2x) - 4B\sin(2x)) - 2(-2A\sin(2x) + 2B\cos(2x))$

$+ (A\cos(2x) + B\sin(2x)) = 3\cos(2x)$

$(-4A - 4B + A)\cos(2x) + (-4B + 4A + B)\sin 2x = 3\cos(2x) + 0\sin(2x)$

$-3A - 4B = 3$

$-3B + 4A = 0$

$A = -\frac{9}{25}, \quad B = -\frac{12}{25}$

$$z = -\frac{9}{25}\cos(2x) - \frac{12}{25}\sin(2x)$$

general: $$y = C_1 e^x + C_2 x e^x - \frac{9}{25}\cos(2x) - \frac{12}{25}\sin(2x)$$

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3. Find a particular solution of $y'' - 2y' + 5y = 2\cos(3x) - 4\sin(3x) + e^{2x}$

$\lambda^2 - 2\lambda + 5 = 0 \rightarrow \lambda = 1 \pm 2i$

$y_1 = e^x \cos 2x$

$y_2 = e^x \sin 2x$

$z = A\cos(3x) + B\sin(3x) + Ce^{2x}$

$z' = -3A\sin(3x) + 3B\cos(3x) + 2Ce^{2x}$

$z'' = -9A\cos(3x) - 9B\sin(3x) + 4Ce^{2x}$

$(-9A\cos(3x) - 9B\sin(3x) + 4Ce^{2x})$

$-2(-3A\sin(3x) + 3B\cos(3x) + 2Ce^{2x})$

$+5(A\cos(3x) + B\sin(3x) + Ce^{2x})$

$= 2\cos(3x) - 4\sin(3x) + e^{2x}$

$$(-9A-6B+5A)\cos(3x)+(-9B+6A+5B)\sin(3x)$$

$$+(4C-4C+5C)e^{2x}=2\cos(3x)-4\sin(3x)+e^{2x}$$

$$-9A-6B+5A=2$$

$$-4A-6B=2$$

$$4A+6B=-2$$

$$-9B+6A+5B=-4$$

$$6A-4B=-4$$

$$4C-4C+5C=1$$

$$5C=1\rightarrow C=\frac{1}{5}$$

$$4A + 6B = -2$$
$$6A - 4B = -4$$

$\Big\}$ $A = -\frac{8}{13}$, $B = \frac{1}{13}$

$$z = -\frac{8}{13}\cos(3x) + \frac{1}{13}\sin(3x) + \frac{1}{5}e^{2x}$$

General:

$$y = C_1 e^{x}\cos(2x) + C_2 e^{x}\sin(2x) - \frac{8}{13}\cos(3x) + \frac{1}{13}\sin(3x) + \frac{1}{5}e^{2x}$$

PopOS: CCCCCC

$\lambda^2 + 9 = 0 \rightarrow \lambda = \pm 3i = 0 \pm 3i$

4. Find a particular solution of $y'' + 9y = 4e^x \sin(2x)$

$\{\sin(3x), \cos(3x)\}$

$f(x) = 4e^x \sin(2x)$

$z = Ae^x \sin(2x) + Be^x \cos(2x)$

$z' = Ae^x \sin(2x) + 2Ae^x \cos(2x) + Be^x \cos(2x)$
$\quad - 2Be^x \sin(2x)$

$= (A - 2B)e^x \sin(2x) + (2A + B)e^x \cos(2x)$

$z'' = (A - 2B)[e^x \sin(2x) + 2e^x \cos(2x)]$
$\quad + (2A + B)[e^x \cos(2x) - 2e^x \sin(2x)]$

$$= \left[(A-2B) - 2(2A+B)\right]e^x \sin(2x)$$

$$+\left[2(A-2B) + (2A+B)\right] e^x \cos(2x)$$

$$= (-3A-4B)e^x \sin(2x) + (4A-3B)e^x \cos(2x)$$

$$y'' + 9y = 4e^x \sin 2x$$

$$(-3A-4B)e^x \sin(2x) + (4A-3B)e^x \cos(2x)$$

$$+ 9\left(Ae^x \sin(2x) + Be^x \cos(2x)\right)$$

$$= 4e^x \sin 2x$$

$$(6A-4B)e^x \sin(2x) + (4A+6B)e^x \cos(2x) = 4e^x \sin 2x$$

$$6A - 4B = 4$$
$$4A + 6B = 0$$
$$\rightarrow A = \frac{6}{13},\ B = \frac{-4}{13}$$

$$z = \frac{6}{13} e^{x} \sin(2x) - \frac{4}{13} e^{x} \cos(2x)$$

general: $y = C_1 \sin(3x) + C_2 \cos(3x) + \frac{6}{13} e^{x} \sin(2x)$

$$- \frac{4}{13} e^{x} \cos(2x)$$

We do have a problem though if our trial solution for *z* is also a solution of the reduced equation.

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	try $z(x) =$*
ce^{rx}	Ae^{rx}
$c \cos \beta x + d \sin \beta x$	$z(x) = A \cos \beta x + B \sin \beta x$
$ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x$	$z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$

***Note:** If z satisfies the reduced equation, try xz; if xz also satisfies the reduced equation, then $x^2 z$ will give a particular solution

5. Find a particular solution of $y'' + y' - 6y = 3e^{2x}$

$\lambda^2 + \lambda - 6 = 0 \to \lambda = -3, 2$

$\{e^{-3x}, e^{2x}\}$

$f(x) = 3e^{2x}$

if we plug in Ae^{2x}, we get 0, not $3e^{2x}$

$z = Axe^{2x}$

$z' = Ae^{2x} + 2Axe^{2x}$

$z'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

$= 4Ae^{2x} + 4Axe^{2x}$

$(4Ae^{2x} + 4Axe^{2x}) + (Ae^{2x} + 2Axe^{2x}) - 6(Axe^{2x})$

$= 3e^{2x}$

$$(4+A)e^{2x} + (4A + 2A - 6A)xe^{2x} = 3e^{2x}$$

$$0$$

$$(4+A)e^{2x} = 3e^{2x}$$

$$A = -1$$

$$z = -xe^{2x}$$

$$\boxed{y = -xe^{2x} + C_1 e^{-3x} + C_2 e^{2x}}$$

The general version of the method of undetermined coefficients can be summarized as follows:

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) = p(x)e^{rx}$ (where $p(x)$ is a polynomial of degree n)

Then try $z(x) = (A_0 + A_1x + A_2x^2 + \cdots + A_nx^n)e^{rx}$

If $f(x) = p_1(x)\cos(\beta x) + p_2(x)\sin(\beta x)$ (where p_1(x) has degree k and p_2 has degree m, let n=max(k, m).

Then try $z(x) = (A_0 + A_1x + \cdots + A_nx^n)\cos(\beta x) + (B_0 + B_1x + \cdots + B_nx^n)\sin(\beta x)$

If $f(x) = p_1(x)e^{\alpha x}\cos(\beta x) + p_2(x)e^{\alpha x}\sin(\beta x)$ (where p_1(x) has degree k and p_2 has degree m, let n=max(k, m).

Then try $z(x) = (A_0 + A_1x + \cdots + A_nx^n)e^{\alpha x}\cos(\beta x) + (B_0 + B_1x + \cdots + B_nx^n)e^{\alpha x}\sin(\beta x)$

NOTE: If any term z satisfies the reduced equation $y'' + ay' + by = 0$ then use xz as the trial solution; if any term xz satisfies the reduced equation, then x^2z will give a particular solution.

$\lambda^2 - \lambda - 6 = 0 \rightarrow \lambda = 3, -2$

6. $y'' - y' - 6y = (2x^2 - 1)e^{2x}$

$\{e^{3x}, e^{-2x}\}$

$f(x) = (2x^2 - 1)e^{2x}$

$p(x)e^{2x}$, where $p(x)$ has degree 2

Generic Polynomial of degree 2: $Ax^2 + Bx + C$

$z = (Ax^2 + Bx + C)e^{2x}$

$z' = 2(Ax^2 + Bx + C)e^{2x} + (2Ax + B)e^{2x}$

$z'' = 4(Ax^2 + Bx + C)e^{2x} + 2(2Ax + B)e^{2x}$

$+ 2(2Ax + B)e^{2x} + 2Ae^{2x}$

$$z = (Ax^2 + Bx + C)e^{2x}$$

$$z' = (2Ax^2 + (2A+2B)x + B + 2C)e^{2x}$$

$$z'' = (4Ax^2 + (4B + 4A + 4A)x + (4C + 2B + 2B + 2A)e^{2x})$$

$$= (4Ax^2 + (8A + 4B)x + (2A + 4B + 4C))e^{2x}$$

$$y'' - y' - 6y = (2x^2 - 1)e^{2x}$$

$$(4Ax^2 + (8A+4B)x + (2A+4B+4C))e^{2x}$$
$$- (2Ax^2 + (2A+2B)x + (B+2C))e^{2x}$$
$$- (6Ax^2 + 6Bx + 6C) = (2x^2 - 1)e^{2x}$$

$$\left(-4Ax^2 + (6A - 4B)x + (2A + 3B - 4C)\right)e^{2x}$$
$$= (2x^2 - 1)e^{2x}$$

$$-4A = 2 \rightarrow A = -\frac{1}{2}$$
$$6A - 4B = 0 \rightarrow -3 - 4B = 0 \rightarrow B = -\frac{3}{4}$$
$$2A + 3B - 4C = -1 \rightarrow -1 - \frac{9}{4} - 4C = -1$$
$$\rightarrow C = -\frac{9}{16}$$

$$z = \left(-\frac{1}{2}x^2 - \frac{3}{4}x - \frac{9}{16}\right)e^{2x}$$

$$y = C_1 e^{3x} + C_2 e^{-2x} - \left(\frac{1}{2}x^2 + \frac{3}{4}x + \frac{9}{16}\right)e^{2x}$$

$\lambda^2 - 2\lambda - 3 = 0 \rightarrow (\lambda - 3)(\lambda + 1) = 0$

7. $y'' - 2y' - 3y = x\cos(2x) + (3x-1)\sin(2x)$

1st deg. poly

$\lambda = -1, 3$

$y_1 = e^{-x}, \; y_2 = e^{3x}$

$f(x) = x\cos(2x) + (3x-1)\sin 2x$

$z = (Ax + B)\cos 2x + (Cx + D)\sin 2x$

$z' = -2(Ax + B)\sin 2x + A\cos 2x$

$+ 2(Cx + D)\cos 2x + C\sin 2x$

$= (C - 2Ax - 2B)\sin 2x + (A + 2Cx + 2D)\cos 2x$

$z'' = 2(C - 2Ax - 2B)\cos 2x - 2A\sin 2x$

$- 2(A + 2Cx + 2D)\sin 2x + 2C\cos 2x$

$$z'' = (4C - 4Ax - 4B)\cos 2x + (-4A - 4Cx - 4D)\sin 2x$$

Solve

$$z'' - 2z' - 3z = x\cos 2x + (3x-1)\sin 2x$$

to find A, B, C, D

$$\rightarrow A = \frac{1}{13},\ B = -\frac{46}{169},\ C = -\frac{5}{13},\ D = \frac{9}{169}$$

$$z = \left(\frac{1}{13}x - \frac{46}{169}\right)\cos(2x) + \left(-\frac{5}{13}x + \frac{9}{169}\right)\sin(2x)$$

$$y = C_1 e^{-x} + C_2 e^{3x} + \left(\frac{1}{15}x - \frac{46}{169}\right)\cos(2x) + \left(-\frac{5}{13}x + \frac{9}{169}\right)\sin(2x)$$

$\lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$

8. $y'' + 4y = 2xe^x \cos x$

$y_1 = \cos 2x \qquad y_2 = \sin 2x$

$f(x) = 2x\, e^x \cos x$

└ 1st degree

$z = (Ax + B)e^x \cos x + (Cx + D)\, e^x \sin x$

Solve $z'' + 4z = 2xe^x \cos x$

⋮

$A = \frac{2}{5}, \quad B = -\frac{7}{25}, \quad C = \frac{1}{5}, \quad D = -\frac{1}{25}$

$$y = C_1 \cos 2x + C_2 \sin 2x + \left(\frac{2}{5}x - \frac{7}{25}\right)e^x \cos x + \left(\frac{1}{5}x - \frac{1}{25}\right)e^x \sin x$$

$\lambda^2 - 4\lambda - 5 = 0 \rightarrow (\lambda - 5)(\lambda + 1) = 0$

9. Give a form of the particular solution of the differential equation:

a. $y'' - 4y' - 5y = 2\cos(3x) - 5e^{5x} + 4$

$y_1 = e^{5x}$, $y_2 = e^{-x}$

$f(x) = 2\cos(3x) - 5e^{5x} + 4$ → 0-deg. poly.

$z = A\cos 3x + B\sin 3x + Ce^{5x} + D$

WRONG!

$z = A\cos 3x + B\sin 3x + Cxe^{5x} + D$

$\to \lambda^2 + 8\lambda + 16 = 0 \to (\lambda+4)^2 = 0 \Rightarrow \lambda = -4$

b. $y'' + 8y' + 16y = 2x - 1 + 7e^{-4x}$

$y_1 = e^{-4x}$, $y_2 = xe^{-4x}$

$f(x) = 2x - 1 + 7e^{-4x}$

$z = Ax + B + Ce^{-4x}$

$z = Ax + B + Cxe^{-4x}$

$$z = Ax + B + Cx^2e^{-4x}$$

c. $y'' + y = 4\sin(x) - \cos(2x) + 2e^{2x}$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \rightarrow y_1 = \cos x,\ y_2 = \sin x$

$z = A\sin x + B\cos x + C\sin 2x + D\cos 2x + Ee^{2x}$

$$z = Ax\sin x + Bx\sin x + C\sin 2x + D\cos 2x + Ee^{2x}$$

Two Techniques for Nonhomogeneous
2nd-Order Linear Differential Equations

Variation of Parameters (Lecture 9)

$$z = u y_1 + v y_2$$

$$u = \int -\frac{y_2}{W} f(x)\,dx \quad , \quad v = \int \frac{y_1}{W} f(x)\,dx$$

Undetermined Coefficients (Lecture 10)