

9.4

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9.4 Comparisons of Series

- Use the **Direct Comparison Test** to determine whether a series converges or diverges.
- Use the **Limit Comparison Test** to determine whether a series converges or diverges.

Direct Comparison Test

For the convergence tests developed so far, the terms of the series have to be fairly simple and the series must have special characteristics in order for the convergence tests to be applied. A slight deviation from these special characteristics can make a test nonapplicable. For example, in the pairs listed below, the second series cannot be tested by the same convergence test as the first series, even though it is similar to the first.

1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric, but $\sum_{n=0}^{\infty} \frac{n}{2^n}$ is not.
2. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p -series, but $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$ is not.
3. $a_n = \frac{n}{(n^2 + 3)^2}$ is easily integrated, but $b_n = \frac{n^2}{(n^2 + 3)^2}$ is not.

In this section, you will study two additional tests for positive-term series. These two tests greatly expand the variety of series you are able to test for convergence or divergence. They allow you to *compare* a series having complicated terms with a simpler series whose convergence or divergence is known.

REMARK As stated, the Direct Comparison Test requires that $0 < a_n \leq b_n$ for all n . Because the convergence of a series is not dependent on its first several terms, you could modify the test to require only that $0 < a_n \leq b_n$ for all n greater than some integer N .

THEOREM 9.12 Direct Comparison Test™

Let $0 < a_n \leq b_n$ for all n .

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof To prove the first property, let $L = \sum_{n=1}^{\infty} b_n$ and let

$$S_n = a_1 + a_2 + \cdots + a_n.$$

Because $0 < a_n \leq b_n$, the sequence S_1, S_2, S_3, \dots is nondecreasing and bounded above by L ; so, it must converge. Because

$$\lim_{n \rightarrow \infty} S_n = \sum_{n=1}^{\infty} a_n$$

it follows that $\sum_{n=1}^{\infty} a_n$ converges. The second property is logically equivalent to the first.

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

■ **FOR FURTHER INFORMATION** Is the Direct Comparison Test just for nonnegative series? To read about the generalization of this test to real series, see the article “The Comparison Test—Not Just for Nonnegative Series” by Michele Longo and Vincenzo Valori in *Mathematics Magazine*. To view this article, go to MathArticles.com.