

9.3

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9.3 The Integral Test and p -Series

- Use the Integral Test to determine whether an infinite series converges or diverges.
- Use properties of p -series and harmonic series.

The Integral Test

In this and the next section, you will study several convergence tests that apply to series with *positive* terms.

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x) \, dx$$

either both converge or both diverge.

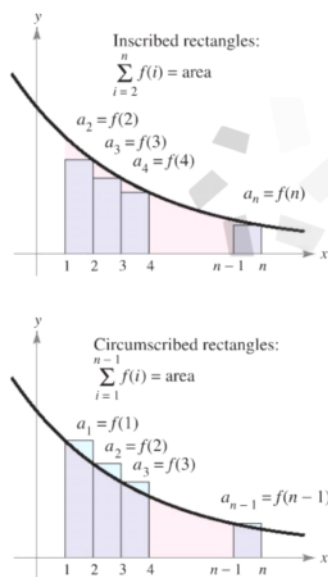


Figure 9.8

Proof Begin by partitioning the interval $[1, n]$ into $(n - 1)$ unit intervals, as shown in Figure 9.8. The total areas of the inscribed rectangles and the circumscribed rectangles are

$$\sum_{i=2}^n f(i) = f(2) + f(3) + \cdots + f(n) \quad \text{Inscribed area}$$

and

$$\sum_{i=1}^{n-1} f(i) = f(1) + f(2) + \cdots + f(n-1) \quad \text{Circumscribed area}$$

The exact area under the graph of f from $x = 1$ to $x = n$ lies between the inscribed and circumscribed areas.

$$\sum_{i=2}^n f(i) \leq \int_1^n f(x) \, dx \leq \sum_{i=1}^{n-1} f(i)$$

Using the n th partial sum, $S_n = f(1) + f(2) + \cdots + f(n)$, you can write this inequality as

$$S_n - f(1) \leq \int_1^n f(x) \, dx \leq S_{n-1}.$$

Now, assuming that $\int_1^{\infty} f(x) \, dx$ converges to L , it follows that for $n \geq 1$

$$S_n - f(1) \leq L \implies S_n \leq L + f(1).$$

Consequently, $\{S_n\}$ is bounded and monotonic, and by Theorem 9.5 it converges. So, $\sum a_n$ converges. For the other direction of the proof, assume that the improper integral diverges. Then $\int_1^n f(x) \, dx$ approaches infinity as $n \rightarrow \infty$, and the inequality $S_{n-1} \geq \int_1^n f(x) \, dx$ implies that $\{S_n\}$ diverges. So, $\sum a_n$ diverges.

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

Remember that the convergence or divergence of $\sum a_n$ is not affected by deleting the first N terms. Similarly, when the conditions for the Integral Test are satisfied for all $x \geq N > 1$, you can simply use the integral $\int_N^{\infty} f(x) \, dx$ to test for convergence or divergence. (This is illustrated in Example 4.)