

9.10

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9.10 Taylor and Maclaurin Series

- Find a Taylor or Maclaurin series for a function.
- Find a binomial series.
- Use a basic list of Taylor series to find other Taylor series.

Taylor Series and Maclaurin Series

In Section 9.9, you derived power series for several functions using geometric series with term-by-term differentiation or integration. In this section, you will study a *general* procedure for deriving the power series for a function that has derivatives of all orders. The next theorem gives the form that *every* convergent power series must take.

• **REMARK** Be sure you understand Theorem 9.22. The theorem says that *if a power series converges to $f(x)$* , then the series must be a Taylor series. The theorem does *not* say that every series formed with the Taylor coefficients $a_n = f^{(n)}(c)/n!$ will converge to $f(x)$.

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n(x - c)^n$ for all x in an open interval I containing c , then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

Proof Consider a power series $\sum a_n(x - c)^n$ that has a radius of convergence R . Then, by Theorem 9.21, you know that the n th derivative of f exists for $|x - c| < R$, and by successive differentiation you obtain the following.

$$\begin{aligned} f^{(0)}(x) &= a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + a_4(x - c)^4 + \cdots \\ f^{(1)}(x) &= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + 4a_4(x - c)^3 + \cdots \\ f^{(2)}(x) &= 2a_2 + 3!a_3(x - c) + 4 \cdot 3a_4(x - c)^2 + \cdots \\ f^{(3)}(x) &= 3!a_3 + 4!a_4(x - c) + \cdots \\ &\vdots \\ f^{(n)}(x) &= n!a_n + (n + 1)!a_{n+1}(x - c) + \cdots \end{aligned}$$

Evaluating each of these derivatives at $x = c$ yields

$$\begin{aligned} f^{(0)}(c) &= 0!a_0 \\ f^{(1)}(c) &= 1!a_1 \\ f^{(2)}(c) &= 2!a_2 \\ f^{(3)}(c) &= 3!a_3 \end{aligned}$$


and, in general, $f^{(n)}(c) = n!a_n$. By solving for a_n , you find that the coefficients of the power series representation of $f(x)$ are

$$a_n = \frac{f^{(n)}(c)}{n!}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof. ■

Notice that the coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c as defined in Section 9.7. For this reason, the series is called the **Taylor series** for $f(x)$ at c .

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COLIN MACLAURIN (1698–1746)
 The development of power series to represent functions is credited to the combined work of many seventeenth- and eighteenth-century mathematicians. Gregory, Newton, John and James Bernoulli, Leibniz, Euler, Lagrange, Wallis, and Fourier all contributed to this work. However, the two names that are most commonly associated with power series are Brook Taylor (1685–1731) and Colin Maclaurin.
 See LarsonCalculus.com to read more of this biography.