1) Use substitution to evaluate $\int_0^{\pi/4} \tan^3(x) \cdot sec^2(x)\, dx$
$A)\, 0 \quad B)\, 0.15 \quad C)\, 0.25 \quad D)\, 0.35 \quad E)\, 0.45 \quad F)\, 0.55 \quad G)\, 0.65 \quad H)\, 0.75$ I) $0.85 \quad J)\, 1$

solution: $(u = tan(x),\; du = sec^2(x)\, dx) \; = \int_0^1 u^3 du = \frac{u^4}{4}\Big|_0^1 \; = \frac{1}{4}\,.\; (C)$

---

2) Use integration by parts to evaluate $\int_1^{e} x^2 ln(x)\, dx.$
$A)4.08 \;\; B)4.16 \;\; C)4.28 \;\; D)4.34 \;\; E)\, 4.42 \;\; F)\, 4.57 \;\; G)\, 4.69 \;\; H)\; 4.74 \;\; I)\; 4.82 \;\; J)\; 4.93$

solution: $(u = ln(x),\, dv = x^2 dx,\; du = \frac{1}{x}dx,\, v = \frac{1}{3}x^3) \;\; = \; \frac{1}{3}x^3 ln(x) - \frac{1}{3}\int x^2 dx =$
$\frac{1}{3}x^3 ln(x) - \frac{1}{9}x^3\,\Big|_1^e \;\; = (\frac{e^3}{3} - \frac{e^3}{9}) - (\, - \frac{1}{9}) = \frac{2e^3+1}{9} \sim 4.57. \;\; (F)$

---

3) Using partial fractions, find a solution to $\int \frac{-1}{x^2 - x}\, dx. \;\; (x > 1)$
A) $arctan(2x - 1)$ B) $(x^2 - x)^{-2}$ C) $ln(x^2 - x)$ D) $ln(x) + ln(x - 1)$ E) $ln(\frac{x}{x-1})$
F) $\frac{ln(x)}{ln(x-1)}$ G) $ln(\frac{x}{\mathrm{x}-1}) + x \;\; H)\, ln(\frac{x}{x-1}) - \frac{1}{x} \quad I)\; \frac{ln(x-1)}{x^2} \quad J)\; \frac{x^2}{ln(x-1)}$

solution: $= \int \frac{1}{x} - \frac{1}{x-1}\; dx$ (partial fractions) $= ln(x) - ln(x - 1) = ln(\frac{x}{x-1}).\;\; (E)$

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4) Find what becomes of the integral $\int \frac{x^2}{\sqrt{9-x^2}}\; dx,$ when you make the substitution
$x = \; 3\, sin(\theta), \;\; - \frac{\pi}{2} \; < \theta < \frac{\pi}{2}.$
A) $3\int sin(\theta)\, d\theta$ B)$3\int cos(\theta)\, d\theta$ C) $\int \frac{9}{sin(\theta)}\, d\theta$ D) $\int \frac{9}{cos(\theta)}\, d\theta$ E) $3\int sec^2(\theta)\, d\theta$
F) $3\int csc^2(\theta)\, d\theta$ G) $9\int sin^2(\theta)\, d\theta$ H) $9\int cos^2(\theta)\, d\theta$ I) $\int \sqrt{9 - sin^2(\theta)}\;\, d\theta$
J) $\int \sqrt{9 - cos^2(\theta)}\;\, d\theta$

solution: $(x = \; 3\, sin(\theta),\; dx = \; 3\, cos(\theta)\, d\theta) \;\; = \int \frac{9\; sin^2(\theta)}{\sqrt{9 - 9\, sin^2(\theta)}}\; 3\, cos(\theta)\, d\theta =$
$\int \frac{9\; sin^2(\theta)}{3\, cos(\theta)}\; 3\, cos(\theta)\, d\theta = \; \int 9\;\, sin^2(\theta)\, d\theta. \;\; (G)$

---

5)Find the **area** of the region enclosed by the curve $y = 2 - x^2$ and the line $y = 2 - 2\, x.$
$A)\, 1 \;\; B)\, \frac{1}{2} \;\; C)\, \frac{3}{2} \quad D)\, \frac{4}{3} \quad E)\, \frac{7}{5} \quad F)\, \frac{5}{3} \quad G)\, \frac{11}{5} \;\; H)\, \frac{9}{2} \;\; I)\, \frac{14}{3} \quad J)\, \frac{13}{5}$

solution: $2 - x^2 = 2 - 2x \Rightarrow x^2 = 2x \Rightarrow \; x = 0\, and\, x = 2$ (intersection points).
$A = \int_0^2 (\, 2 - x^2) - (2 - 2x)\, dx \;\; = \int_0^2 2x - x^2 \; dx \;\; = x^2 - \frac{x^3}{3}\,\Big|_0^2 \; = \frac{4}{3}.\; (D)$

6) Find the **volume** of the solid obtained by **rotating** the region enclosed by the curve $x = \frac{2}{y}$, the lines $y = 1$, $y = 4$, and the y-axis, about the **y-axis**.

$A)\ \frac{\pi}{4}\quad B)\ \frac{3\pi}{4}\quad C)\ \pi\quad D)\ \frac{5\pi}{4}\quad E)\ \frac{7\pi}{4}\quad F)\ \frac{9\pi}{2}\quad G)\ 2\pi\quad H)\ \frac{9\pi}{4}\quad I)\ 3\pi\quad J)\ \frac{4\pi}{3}$

<u>solution</u>: $V = \pi\int_1^4 (\frac{2}{y})^2 dy = \pi\int_1^4 \frac{4}{y^2}\, dy = \pi(-\frac{4}{y})\Big|_1^4 = 3\pi.\ (I)$

---

7) Find the **arc length** of the curve $x = \frac{2}{3}y^{3/2}$, $0 \le y \le 3$.

$A)\ \frac{4}{3}\quad B)\ \frac{8}{3}\quad C)\ \frac{14}{3}\quad D)\ \frac{3}{2}\quad E)\ \frac{7}{2}\quad$ F) $\frac{9}{2}\quad G)\ \frac{10}{3}\quad H)\ \frac{11}{2}\quad I)\ \frac{20}{3}\quad$ J) $\frac{17}{2}$

<u>solution</u>: $\frac{dx}{dy} = y^{\frac{1}{2}}$, $(\frac{dx}{dy})^2 = y$. $L = \int_0^3 \sqrt{1+y}\ dy = \frac{2}{3}(1+y)^{\frac{3}{2}}\Big|_0^3 = \frac{14}{3}.\ (C)$

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8) Find the value of the improper integral $\int_0^\infty \frac{x}{(x^2+2)^2}\, dx$, if it converges.

$A)\ 0$ B) 2 C) 4 D) 6 $E)\ 8\quad F)\ \frac{1}{6}\quad G)\ \frac{1}{4}\quad H)\ \frac{1}{3}$ I) $\frac{2}{3}\quad J)\ diverges$

<u>solution</u>: $(u = x^2 + 2,\ du = 2x\, dx)\ = \frac{1}{2}\int_2^\infty \frac{1}{u^2}du = -\frac{1}{2u}\Big|_2^\infty = \frac{1}{4}.\ (G)$

---

9) Find the solution to the initial value differential equation $\frac{dy}{dt} = t^2 y,\ y(0) = 2.$

$A)\ y = 2\, e^{t^2/2}\quad B)\ y = 2\, e^{3t^2}\quad C)\ y = 2\, e^{t^3/3}\quad D)\ y = (t + \sqrt{2})^2$
$E)\ y = ln(t^2 + 1) + 2\quad F)\ y = 2\, ln(t + e)\quad G)\ y = (t + 1)^{-2} + 2$
$H)\ y = (t + 2)\quad I)\ y = t^2 + \frac{1}{2}\quad J)\ y = \sqrt{t + 4}$

<u>solution</u>: $\int \frac{1}{y}\, dy = \int t^2 dt \Rightarrow ln(y) = \frac{1}{3}t^3 + C \Rightarrow y = Ke^{t^3/3}\ (K = e^C).$
$y(0) = 2 \Rightarrow 2 = Ke^0 = K \Rightarrow\ y = 2\ e^{t^3/3}.\ (C)$

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10) Find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n}.$

$A)\ \frac{5}{3}\quad B)\ \frac{5}{4}\quad C)\ \frac{3}{2}\quad D)\ \frac{9}{5}\quad E)\ \frac{1}{5}\quad F)\ \frac{5}{2}\quad G)\ \frac{2}{3}\quad H)\ \frac{1}{2}\quad I)\ \frac{1}{4}\quad J)\ \frac{2}{5}$

<u>solution</u>: $= \sum_{n=1}^{\infty} \frac{2^n}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{2/3}{1-\frac{2}{3}} - \frac{1/3}{1-\frac{1}{3}} = 2 - \frac{1}{2} = \frac{3}{2}.\ (C)$

11) Approximate the sum $\sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{n!}$ with an error of less than 2 x 10^{-3}.

$A)\,0.367 \quad B)\,0.371 \quad C)\,0.384 \quad D)\,0.396 \quad E)\,0.408 \quad F)\,0.414 \quad G)\,0.429$
$H)\,0.437 \quad I)\,0.4458 \quad J)\,0.452$

<u>solution</u>: $= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \ldots\ldots$. $\frac{1}{5!} = .0083 > \frac{1}{500}$, $\frac{1}{6!} = .001388 < \frac{1}{500}$. Approx $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \sim 0.367.$ (A)

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12) Find the complete <u>interval of convergence</u> (either absolute or conditional) of the power series $\sum\limits_{n=1}^{\infty} \frac{2^n (x-4)^n}{n \cdot 3^n}$.

A) $\frac{5}{2} < x < \frac{11}{2}$ $B)\,\frac{5}{2} \le x < \frac{11}{2}$ $C)\,\frac{5}{2} < x \le \frac{11}{2}$ $D)\,\frac{5}{2} \le x \le \frac{11}{2}$ $E)\,\frac{3}{2} < x < \frac{13}{2}$
$F)\,\frac{3}{2} \le x < \frac{13}{2}$ $G)\,\frac{3}{2} < x \le \frac{13}{2}$ $H)\,\frac{3}{2} \le x \le \frac{13}{2}$ $I)\,-\infty < x < \infty$ $J)\,\{4\}$

<u>solution</u>: Ratio Test : $\frac{2^{n+1}|x-4|^{n+1}}{(n+1)\,3^{n+1}} \; \frac{n\,3^n}{2^n |x-4|^n} = (\frac{n}{n+1})(\frac{2}{3})|x-4| \to \frac{2}{3}|x-4| < 1 \Rightarrow$ $|x-4| < \frac{3}{2} \Rightarrow \frac{5}{2} < x < \frac{11}{2}$. <u>Endpoints</u>: $(x = \frac{11}{2})$ $\sum\limits_{n=1}^{\infty} \frac{2^n(\frac{3}{2})^n}{n \cdot 3^n} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ (diverges) ; (x=$\frac{5}{2}$) $\sum\limits_{n=1}^{\infty} \frac{2^n(-\frac{3}{2})^n}{n \cdot 3^n} = \sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n}$ (converges). Therefore interval is $\frac{5}{2} \le x < \frac{11}{2}$. (B)

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13) Find the Taylor polynomial of order 2 for $f(x) = \frac{1}{x}$ at $x = 1$.

$A) 1 + (x-1) + (x-1)^2 \quad B) 1 - (x-1) + (x-1)^2 \quad C) 1 + (x-1) - (x-1)^2$
$D) 1 + \frac{1}{2}(x-1) + (x-1)^2 \quad E) 1 - (x-1) + \frac{1}{2}(x-1)^2 \quad F) 1 - \frac{1}{2}(x-1) + (x-1)^2$
$G)\, 1 + \frac{1}{2}(x-1) + \frac{3}{2}(x-1)^2 \quad H)\, 1 - \frac{1}{2}(x-1) + \frac{3}{2}(x-1)^2$
$I)\, 1 + \frac{1}{2}(x-1) - \frac{3}{2}(x-1)^2 \quad J)\, 1 - \frac{1}{2}(x-1) - \frac{3}{2}(x-1)^2\,|$

<u>solution</u> : $f(1) = 1,\ c_0 = 1;\ f'(x) = -\frac{1}{x^2}, f'(1) = -1,\ c_1 = -1;\ f''(x) = \frac{2}{x^3}$, $f''(1) = 2,\ c_2 = \frac{2}{2!} = 1.$ $\quad P_2(x) = 1 - (x-1) + (x-1)^2.$ (B)

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14) If the Maclaurin series for $x \cdot e^{2x}$ is $\sum\limits_{n=1}^{\infty} c_n\, x^n$, then find c_4.

$A)\ \frac{1}{2} \quad B)\ \frac{3}{2} \quad C)\ \frac{2}{3} \quad D)\ \frac{4}{3} \quad E)\ \frac{1}{4} \quad F)\ \frac{3}{4} \quad G)\ \frac{5}{2} \quad H)\ \frac{5}{3} \quad I)\ \frac{5}{6} \quad J)\ \frac{6}{5}$

<u>solution</u>: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$.
$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \ldots$.
$x\,e^{2x} = x + 2x^2 + 2x^3 + \frac{4}{3}x^4 + \frac{2}{3}x^5 + \ldots$. $\quad c_4 = \frac{4}{3}$ (D)

15) Using the Maclaurin series for $ln(1+x)$, approximate $\int_0^{0.5} ln(1+x)\,dx$, with an error $\leq$ 0.002.

$A)$ 0.109375 $B)$ 0.210432 $C)$ 0.342871 $D)$ 0.465732 $E)$ 0.543876
$F)$ 0.623149 $G)$ 0.743987 $H)$ 0.841762 $I)$ 0.954129 $J)$ 1.097658

<u>solution</u>: $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ \ldots$ $\int_0^{0.5} ln(1+x)\,dx =$
$\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^6}{30} - \ldots. \Big|_0^{0.5} = \frac{(0.5)^2}{2} - \frac{(0.5)^3}{6} + \frac{(0.5)^4}{12} - \frac{(0.5)^5}{20} + \frac{(0.5)^6}{30} - \ldots.$.
$\frac{(0.5)^5}{20} < 0.002$, $\frac{(0.5)^4}{12} > 0.002$. $\int_0^{0.5} ln(1+x)\,dx \sim \frac{(0.5)^2}{2} - \frac{(0.5)^3}{6} + \frac{(0.5)^4}{12} = .109375\,(A)$

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16) Find the Maclaurin series for the function $x^2 - x \cdot tan^{-1}(x)$.

$A)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{2n+2}$ $B)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+4}}{2n+2}$ $C)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{2n+1}$ $D)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+4}}{2n+3}$
$E)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{3n-1}}{3n}$ $F)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{3n+2}}{3n+1}$ $G)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{3n+3}}{3n+2}$ $H)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{3n+4}}{3n+3}$
$I)\ \sum_{n=0}^{\infty} \frac{(-1)^n\, (x^2)^{2n+1}}{4n+1}$ $J)\ \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+2}}{4n+2}$

<u>solution</u>: $tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \ ..\ ;$ $x \cdot tan^{-1}(x) = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} - \ ..$
$x^2 - x \cdot tan^{-1}(x) = \frac{x^4}{3} - \frac{x^6}{5} + \frac{x^8}{7} - \ \ldots. = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+4}}{2n+3}$. (D)

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17) Find the Taylor series for $f(x) = 1 + x + x^2$ centered at $a = 3$.

$A)\ 13 + 5(x-3) + (x-3)^2$ $B)\ 12 + 5(x-3) + 2(x-3)^2$
$C)\ 13 + 6(x-3) + (x-3)^2$ $D)\ 12 + 6(x-3) + 2(x-3)^2$
$E)\ 13 + 7(x-3) + (x-3)^2$ $F)\ 13 + 7(x-3) + 2(x-3)^2$
$G)\ 13 + 8(x-3) + (x-3)^2$ $H)\ 12 + 8(x-3) + 2(x-3)^2$
$I)\ 13 + 9(x-3) + (x-3)^2$ $J)\ 12 + 9(x-3) + 2(x-3)^2$

<u>solution</u> : $f(3) = 13,\ c_0 = 13;\ f'(x) = 1 + 2x,\ f'(3) = 7,\ c_2 = 7;$
$f''(x) = 2,\ c_2 = \frac{2}{2!} = 1,\ f^{(n)}(x) = 0$ for all $n > 2 \Rightarrow c_n = 0$ for all $n > 2$.

Taylor series $= 13 + 7(x-3) + (x-3)^2$. (E)

18) If $sin(x) = \sum_{n=0}^{\infty} c_n (x - \frac{\pi}{4})^n$, is the Taylor series for the function $sin(x)$, centered at $a = \frac{\pi}{4}$, then find the coeficient, c_2, of $(x - \frac{\pi}{4})^2$.

A) $-\frac{\sqrt{2}}{4}$ $B)\ \frac{1}{6}$ $C)\ -\frac{\sqrt{2}}{8}$ $D)\ \frac{1}{10}$ $E)\ -\frac{\sqrt{2}}{12}$ $F)\ \frac{1}{14}$ $G)\ -\frac{\sqrt{2}}{16}$
$H)\ \frac{1}{18}$ $I)\ -\frac{\sqrt{2}}{20}$ $J)\ \frac{1}{22}$

<u>solution</u>: $c_2 = \frac{f''(\frac{\pi}{4})}{2!}$. $f(x) = sin(x),\ f'(x) = cos(x),\ f''(x) = -sin(x).$
$f''(\frac{\pi}{4}) = -sin(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$. $c_2 = -\frac{\sqrt{2}}{4}$. (A)

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19) From the formula of the Binomial Series, we get that
$(1-2x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} c_n x^n$, with $c_3 =$:

A) 0 B) $\frac{1}{4}$ $C)\ -\frac{1}{4}$ $D)\ \frac{1}{2}$ $E)\ -\frac{1}{2}$ $F)\ \frac{3}{8}$ $G)\ -\frac{3}{8}$ $H)\ \frac{4}{27}$ $I)\ -\frac{4}{27}$ $J)\ 1$

<u>solution</u>: $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6}x^3 + ...$.
$(1-2x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-2x) - \frac{1}{8}(-2x)^2 + \frac{3}{48}(-2x)^3 + ... =$
$1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + ...$. $c_3 = -\frac{1}{2}$ (E)

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20) Using the Alternating Series Estimation Theorem, and the Maclauren Series for e^x, estimate $\int_0^{0.1} e^{-x^2} dx$, with an error less than or equal to 1 x 10^{-6} .

$A)$ .0923367 B) .0932765 C) .0943876 D) .0957627 E) .0963333
F) .0976667 $G)$ .0987433 $H)$ .0996667 $I)$ .1095233 $J)$ .1136667

<u>solution</u>: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$.
$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \frac{(-x^2)^5}{5!} + \ ... =$
$1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + ...$.
$\int_0^{0.1} e^{-x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + ... \Big|_0^{0.1} =$
$0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{10} - \frac{(0.1)^7}{42} + \frac{(0.1)^9}{216} - \frac{(0.1)^{11}}{1320} + ...$.
$\frac{(0.1)^5}{10} = 10^{-6}$, so approximation is $0.1 - \frac{(0.1)^3}{3} = .099666666$ (H)