

- 1) Use substitution to evaluate $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$
 A) 0 B) 1 C) 2 D) 3 E) 4 F) 5 G) 6 H) 7 I) 8 J) 9

solution: ($u = x^2 + 1, dx = 2x dx$) = $\int_1^4 2u^{-\frac{1}{2}} du = 4u^{\frac{1}{2}}|_1^4 = 4(2 - 1) = 4.$ (E)

- 2) Using integration by parts $\int x e^{3x} dx = f(x) + C$, where $f(x) =$:
 A) $\frac{1}{3} x e^{3x}$ B) $3x e^{3x}$ C) $\frac{1}{3} x e^{3x} - \frac{1}{3} e^{3x}$ D) $3x e^{3x} - 3 e^{3x}$ E) $\frac{1}{3} x e^{3x} - 3 e^{3x}$
 F) $3x e^{3x} - \frac{1}{3} e^{3x}$ G) $\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x}$ H) $3x e^{3x} - 9 e^{3x}$ I) $\frac{1}{9} x e^{3x} - \frac{1}{3} e^{3x}$
 J) $\frac{1}{9} x e^{3x} - \frac{1}{3} e^{3x}$

solution: ($u = x, du = dx, dv = e^{3x} dx, v = \frac{1}{3} e^{3x}$) = $\frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} dx =$
 $\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C.$ (G)

- 3) Using partial fractions, find a solution to $\int \frac{1}{(3x-1)(3x+1)} dx.$ ($x > \frac{1}{3}$)
 A) $\frac{1}{2} \ln\left(\frac{3x-1}{3x+1}\right)$ B) $\frac{1}{3} \ln\left(\frac{3x-1}{3x+1}\right)$ C) $\frac{1}{6} \ln\left(\frac{3x-1}{3x+1}\right)$ D) $\frac{1}{2} \ln\left(\frac{3x+1}{3x-1}\right)$ E) $\frac{1}{3} \ln\left(\frac{3x+1}{3x-1}\right)$
 F) $\frac{1}{6} \ln\left(\frac{3x+1}{3x-1}\right)$ G) $\frac{1}{2} \ln[(3x-1)(3x+1)]$ H) $\frac{1}{3} \ln[(3x-1)(3x+1)]$
 I) $\frac{1}{6} \ln[(3x-1)(3x+1)]$ J) $e^{3x-1} * e^{3x+1}$

solution: $\int \frac{1}{(3x-1)(3x+1)} dx = \int \left(\frac{1/2}{(3x-1)} - \frac{1/2}{(3x+1)}\right) dx = \frac{1}{6} \ln(3x-1) - \frac{1}{6} \ln(3x+1) =$
 $\frac{1}{6} \ln\left(\frac{3x-1}{3x+1}\right) + C.$ (C)

- 4) Find what becomes of the integral $\int \frac{27}{(x^2+9)^2} dx$, when you make the substitution
 $x = 3 \tan(\theta), 0 < \theta < \frac{\pi}{2}.$
 A) $\int \sin(\theta) d\theta$ B) $\int \cos(\theta) d\theta$ C) $\int \frac{1}{\sin(\theta)} d\theta$ D) $\int \frac{1}{\cos(\theta)} d\theta$ E) $\int \sec^2(\theta) d\theta$
 F) $\int \csc^2(\theta) d\theta$ G) $\int \sin^2(\theta) d\theta$ H) $\int \cos^2(\theta) d\theta$ I) $\int (\tan^2(\theta) + 9) d\theta$
 J) $\int (\sec^2(\theta) + 9) d\theta$

solution: ($x = 3 \tan(\theta), dx = 3 \sec^2(\theta) d\theta$) = $\int \frac{27 \cdot 3 \sec^2(\theta) d\theta}{(9 \tan^2(\theta) + 9)^2} =$
 $\int \frac{27(3 \sec^2(\theta)) d\theta}{81 \sec^4(\theta)} = \int \frac{1}{\sec^2(\theta)} d\theta = \int \cos^2(\theta) d\theta.$ (H)

5) Find the **area** of the region enclosed by the curve $y = \sqrt{x}$ and the line $x = 2y$ ($y = \frac{1}{2}x$).

A) 1 B) $\frac{1}{2}$ C) $\frac{3}{2}$ D) $\frac{4}{3}$ E) $\frac{7}{5}$ F) $\frac{5}{3}$ G) $\frac{11}{5}$ H) $\frac{9}{2}$ I) $\frac{14}{3}$ J) $\frac{13}{5}$

solution: Intersection points $(0, 0)$ and $(4, 2)$. $y = \sqrt{x}$ is always top, and $y = \frac{1}{2}x$ always bottom. $A = \int_0^4 (x^{\frac{1}{2}} - \frac{1}{2}x) dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2 \Big|_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$. (D)

6) Find the **volume** of the solid obtained by **rotating** the region enclosed by the curve $y = 2\sqrt{x}$, and the lines $y = 2$, and $x = 0$ about the **x-axis**.

A) $\frac{\pi}{4}$ B) $\frac{3\pi}{4}$ C) π D) $\frac{5\pi}{4}$ E) $\frac{7\pi}{4}$ F) $\frac{9\pi}{2}$ G) 2π H) $\frac{9\pi}{4}$ I) 3π J) $\frac{4\pi}{3}$

solution: $y = 2\sqrt{x}$ and $y = 2$ intersect at $(0, 0)$ and $(1, 2)$. Using the washer method we get $V = \pi \int_0^1 (4 - 4x) dx = \pi (4x - 2x^2) \Big|_0^1 = 2\pi$. (G)

7) Find the **arc length** of the curve $x = 2y^{3/2}$, $0 \leq y \leq 1$, to 2 decimal places.

A) 1.57 B) 1.94 C) 2.27 D) 2.84 E) 3.45 F) 3.92 G) 4.46 H) 4.84 I) 5.17 J) 5.68

solution: $\frac{dx}{dy} = 3y^{\frac{1}{2}} \Rightarrow (\frac{dx}{dy})^2 = 9y \Rightarrow L = \int_0^1 \sqrt{1 + 9y} dy$
 $(u = 1 + 9y, du = 9 dy) = \frac{1}{9} \int_1^{10} u^{\frac{1}{2}} du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10} = \frac{2}{27} (10^{3/2} - 1) = 2.268354 \sim 2.27$. (C)

8) Find the value of the improper integral $\int_0^1 \frac{1}{1-x} dx$, if it converges.

A) 0 B) 2 C) 4 D) 6 E) 8 F) $\frac{1}{e}$ G) $\frac{e}{4}$ H) $\frac{1}{3e}$ I) $\frac{2e}{3}$ J) *diverges*

solution: $= \lim_{b \rightarrow 1} (-\ln(1-x) \Big|_0^b) = \lim_{b \rightarrow 1} -\ln(1-b) = \infty$, *diverges*. (J)

9) Find the solution to the differential equation $\frac{dy}{dt} = 3t^2 e^{-y}$, with initial condition $y(0) = \ln(2)$.

A) $y = 2e^{t^2/2}$ B) $y = 2e^{3t^2}$ C) $y = 2e^{t^3/3}$ D) $y = \ln(t^3) + \ln(2)$
 E) $y = \ln(t^3 + 2)$ F) $y = 2\ln(t^3 + e)$ G) $y = (t+1)^{-2} + 2$ H) $y = (t+2)$
 I) $y = t^2 + \frac{1}{2}$ J) $y = \sqrt{t+4}$

solution: $\int e^y dy = \int 3t^2 dt + C \Rightarrow e^y = t^3 + C$. Initial condition is $t = 0, y = \ln(2)$ so $e^{\ln(2)} = C \Rightarrow C = 2$ and we get $e^y = t^3 + 2 \Rightarrow y = \ln(t^3 + 2)$. (E)

10) Find the sum of the infinite series $\sum_{n=1}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right)$.

A) $\frac{5}{3}$ B) $\frac{5}{4}$ C) $\frac{3}{2}$ D) $\frac{9}{2}$ E) $\frac{1}{5}$ F) $\frac{5}{2}$ G) $\frac{2}{3}$ H) $\frac{1}{2}$ I) $\frac{1}{4}$ J) $\frac{2}{5}$

solution: $= \sum_{n=1}^{\infty} \frac{5}{2^n} - \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{5/2}{1-\frac{1}{2}} - \frac{1/3}{1-\frac{1}{3}} = 5 - \frac{1}{2} = \frac{9}{2}$. (D)

11) Using the fewest terms possible, approximate the sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ with an error of less than 0.25.

A) $1/3$ B) $2/3$ C) $1/4$ D) $1/2$ E) $3/4$ F) $1/5$ G) $2/5$ H) $3/5$ I) $4/5$ J) 1

solution: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \dots$ $\frac{4}{17} < 0.25$ and $\frac{3}{10} > 0.25$,
so approximation will be the first 3 terms, $\frac{1}{2} - \frac{2}{5} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5}$. (G)

12) Find the complete interval of convergence (either absolute or conditional) of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n \cdot 2^n}$.

A) $-2 < x < 2$ B) $-2 \leq x < 2$ C) $-2 < x \leq 2$ D) $-2 \leq x \leq 2$
E) $-4 < x < 0$ F) $-4 \leq x < 0$ G) $-4 < x \leq 0$ H) $-4 \leq x \leq 0$
I) $\{0\}$ J) $\{-2\}$

solution: Ratio test $\frac{|(x+2)|^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{|(x+2)|^n} = \frac{1}{2} \frac{n}{n+1} |x+2| \rightarrow \frac{|x+2|}{2} < 1$.
 $|x+2| < 2 \Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0$, $R = 2$, $a = -2$.

Now check endpoints: $x = 0$ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

$x = -4$ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$ diverges, $-4 < x \leq 0$. (G)

13) If $\sum_{n=0}^{\infty} a_n(x-1)^n$ is the Taylor series of $f(x) = \ln(1+x)$ centered at $a = 1$,

then $a_3 =$:

A) $\frac{1}{2}$ B) $-\frac{1}{4}$ C) $\frac{1}{8}$ D) $-\frac{1}{16}$ E) $\frac{1}{24}$ F) $-\frac{1}{30}$ G) $\frac{1}{32}$ H) $-\frac{1}{36}$ I) $\frac{1}{40}$ J) $-\frac{1}{42}$

solution: $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$.

$f'''(1) = \frac{2}{8} = \frac{1}{4}$, $a_3 = \frac{f'''(1)}{3!} = \frac{1}{4} \frac{1}{6} = \frac{1}{24}$. (E)

14) Find the Taylor polynomial of order 6 for $x \cdot \sin(2x)$ at $x = 0$.

- A) $2x^2 - 4x^4 + 8x^6$ B) $2x^2 - \frac{2}{3}x^4 + \frac{5}{2}x^6$ C) $4x^2 - \frac{3}{4}x^4 + \frac{2}{3}x^6$
 D) $4x^2 - \frac{5}{2}x^4 + \frac{2}{15}x^6$ E) $2x^2 - \frac{2}{15}x^4 + \frac{3}{8}x^6$ F) $2x^2 - \frac{5}{16}x^4 + \frac{3}{8}x^6$
 G) $4x^2 - \frac{2}{15}x^4 + \frac{5}{2}x^6$ H) $4x^2 - \frac{3}{8}x^4 + \frac{2}{15}x^6$ I) $2x^2 - \frac{4}{3}x^4 + \frac{4}{15}x^6$
 J) $2x^2 - \frac{1}{5}x^4 + \frac{3}{16}x^6$

solution: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$. $\sin(2x) = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \dots$.
 $x \sin(2x) = 2x^2 - \frac{2^3x^4}{3!} + \frac{2^5x^6}{5!} - \dots$. **So $T_6 = 2x^2 - \frac{4}{3}x^4 + \frac{4}{15}x^6$.** (I)

15) Using the Maclaurin series for $\sin(x^2)$, approximate $\int_0^1 \sin(x^2) dx$,
 with an error ≤ 0.002 .

- A) 0.109375 B) 0.210432 C) 0.309524 D) 0.465732 E) 0.543876
 F) 0.623149 G) 0.743987 H) 0.841762 I) 0.954129 J) 1.097658

solution: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$, so $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$.
 $\int_0^1 \sin(x^2) dx = \int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots dx = \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \frac{x^{15}}{75,600} + \dots \Big|_0^1 =$
 $\frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75,600} + \dots$. $\frac{1}{42} > 0.002$ and $\frac{1}{1320} < 0.002$, so approximation is
 $\frac{1}{3} - \frac{1}{42} = 0.309524$. (C)

16) Find the Maclaurin series for the function $\frac{1 - \cos(x^2)}{x^2}$.

- A) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$ B) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+2)!}$ C) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$ D) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+3)!}$
 E) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n-1}}{3n!}$ F) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{(3n+1)!}$ G) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{(2n+2)!}$ H) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+2)!}$
 I) $\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(4n+1)!}$ J) $\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+2}}{(4n+2)!}$

solution: $\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots$.
 $1 - \cos(x^2) = \frac{x^4}{2!} - \frac{x^8}{4!} + \frac{x^{12}}{6!} - \dots$. $\frac{1 - \cos(x^2)}{x^2} = \frac{x^2}{2!} - \frac{x^6}{4!} + \frac{x^{10}}{6!} - \dots =$
 $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+2)!}$. (H)

17) Find the Taylor series for $f(x) = 2x^3 + x^2 + 3x - 8$ centered at $a = 1$.

- A) $-2 + 9(x-1) + 8(x-1)^2 + 4(x-1)^3$ | **solution:** $f(1) = -2, a_0 = -2$.
 B) $-2 + 6(x-1) + 12(x-1)^2 + (x-1)^3$ | $f'(x) = 6x^2 + 2x + 3, f'(1) = 11$,
 C) $-2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$ | $a_1 = 11, f''(x) = 12x + 2$,
 D) $-2 + 6(x-1) + 4(x-1)^2 + 6(x-1)^3$ | $f''(1) = 14, a_2 = \frac{14}{2!} = 7$.
 E) $-2 + 11(x-1) + 10(x-1)^2 + 4(x-1)^3$ | $f''(x) = 12, f''(1) = 12$,
 F) $-2 + 8(x-1) + 6(x-1)^2 + (x-1)^3$ | $a_3 = \frac{12}{3!} = 2$
 G) $-2 + 11(x-1) + 7(x-1)^2 + 6(x-1)^3$ | *Taylor Series* = $-2 + 11(x-1)$
 H) $-2 + 6(x-1) + 7(x-1)^2 + 2(x-1)^3$ | $+ 7(x-1)^2 + 2(x-1)^3$.
 I) $-2 + 8(x-1) + 12(x-1)^2 + 4(x-1)^3$ | (C)
 J) $-2 + 11(x-1) + 10(x-1)^2 + 8(x-1)^3$
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18) If $\sec(x) = \sum_{n=0}^{\infty} a_n (x - \frac{\pi}{4})^n$ is the Taylor series for the function $\sec(x)$,

centered at $a = \frac{\pi}{4}$, then find the coefficient a_2 .

- A) $\sqrt{2}$ B) $2 + \sqrt{2}$ C) $2\sqrt{2}$ D) $2 + 2\sqrt{2}$ E) $1 + 2\sqrt{2}$
 F) $1 + \sqrt{2}$ G) $\frac{\sqrt{2}}{2}$ H) $\frac{2}{\sqrt{2}}$ I) $\frac{3\sqrt{2}}{2}$ J) $3\sqrt{2}$

solution: $f'(x) = \sec(x) \tan(x), f''(x) = \sec(x) \tan^2(x) + \sec^3(x)$,

$$f''(\frac{\pi}{4}) = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}, a_2 = \frac{f''(\frac{\pi}{4})}{2!} = \frac{3\sqrt{2}}{2}. (I)$$

19) Using the Binomial Series, $(1+x)^{-\frac{1}{3}} = 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{3}}{k} x^k$, find the a_6 term

in the Maclaurin series of $f(x) = \frac{1}{(1-2x^2)^{1/3}} = \sum_{k=0}^{\infty} a_k x^k$.

- A) 16/9 B) -16/9 C) 24/27 D) -24/27 E) 24/9 F) -24/9
 G) 64/81 H) -64/81 I) 112/81 J) -112/81

solution: $(1+x)^{-\frac{1}{3}} = 1 + \binom{-\frac{1}{3}}{1}x + \binom{-\frac{1}{3}}{2}x^2 + \binom{-\frac{1}{3}}{3}x^3 + \dots =$

$$1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \dots \cdot \frac{1}{(1-2x)^{1/3}} = (1-2x)^{-\frac{1}{3}} =$$

$$1 - \frac{1}{3}(-2x^2) + \frac{2}{9}(-2x^2)^2 - \frac{14}{81}(-2x^2)^3 + \dots = 1 + \frac{2}{3}x^2 + \frac{8}{9}x^4 + \frac{112}{81}x^6 + \dots$$

$$\text{So } a_6 = \frac{112}{81}. (I)$$

20) Using the Alternating Series Estimation Theorem, and the Maclaren Series for $\cos(x)$, approximate $\int_0^1 \left(\frac{1-\cos(x)}{x^2}\right) dx$, with an error less than or equal to 1×10^{-5} .

- A) 0.423367 B) 0.432765 C) 0.443876 D) 0.457627 E) 0.463333
 F) 0.476667 G) 0.486389 H) 0.496667 I) 0.515233 J) 0.526667

solution: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$

$$1 - \cos(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots$$

$$\int_0^1 \left(\frac{1-\cos(x)}{x^2}\right) dx = \int_0^1 \left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \dots\right) dx = \frac{x}{2} - \frac{x^3}{72} + \frac{x^5}{3600} - \frac{x^7}{282,240} + \dots \Big|_0^1 =$$

$$\frac{1}{2} - \frac{1}{72} + \frac{1}{3600} - \frac{1}{282,240} + \dots \quad \frac{1}{282,240} \sim 3.54 \times 10^{-6} < 1 \times 10^{-5},$$

$$\frac{1}{3600} \sim 2.78 \times 10^{-4} > 1 \times 10^{-5}. \text{ Approximation is } \frac{1}{2} - \frac{1}{72} + \frac{1}{3600} \sim \mathbf{.486389}$$

(G)