PROBLEM SET

Practice Problems for Exam #1

Math 2350, Fall 2004

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ANSWERS

Problem 1.

The position vector of a particle is given by $\mathbf{R}(t) = (t, t^2, t^3)$. Find the velocity and acceleration vectors of the particle. Find the speed of the particle at $t = 1$.

Answer:
For the velocity vector, we have

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = (1, 2t, 3t^2).$$

For the acceleration vector, we get

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = (0, 2, 6t).$$

The velocity vector at $t = 1$ is

$$\mathbf{V}(1) = (1, 2, 3).$$

The speed at $t = 1$ is

$$\|\mathbf{V}(1)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Problem 2.

The acceleration vector of a particle moving in space is $\mathbf{A}(t) = \mathbf{i} + 2t\mathbf{j} + \sin(t)\mathbf{k}$. The velocity vector of the particle at $t = 0$ is $\mathbf{V}_0 = \mathbf{j} + \mathbf{k}$ and the position vector at time $t = 0$ is $\mathbf{R}_0 = 2\mathbf{i}$. Find $\mathbf{R}(t)$, the position vector at an arbitrary time t.

Answer:
Since $\mathbf{A}(t) = d\mathbf{V}/dt$, we have

$$\begin{aligned}\mathbf{V}(t) &= \int \mathbf{A}(t)\, dt \\ &= \int (\mathbf{i} + 2t\mathbf{j} + \sin(t)\mathbf{k})\, dt \\ &= t\mathbf{i} + t^2\mathbf{j} - \cos(t)\mathbf{k} + \mathbf{C},\end{aligned}$$

where $\mathbf{C}$ is a constant vector. Setting $t = 0$ in this equation gives us

$$\mathbf{V}_0 = \mathbf{V}(0) = -\mathbf{k} + \mathbf{C}.$$

Thus, we have

$$\mathbf{C} = \mathbf{V}_0 + \mathbf{k} = (\mathbf{j} + \mathbf{k}) + \mathbf{k} = \mathbf{j} + 2\mathbf{k}.$$

This gives us

$$\begin{aligned}\mathbf{V}(t) &= t\mathbf{i} + t^2\mathbf{j} - \cos(t)\mathbf{k} + \mathbf{C} \\ &= (t\mathbf{i} + t^2\mathbf{j} - \cos(t)\mathbf{k}) + (\mathbf{j} + 2\mathbf{k}) \\ &= t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - \cos(t))\mathbf{k}.\end{aligned}$$

Since $\mathbf{V}(t) = d\mathbf{R}/dt$, we have

$$\begin{aligned}\mathbf{R}(t) &= \int \mathbf{V}(t)\,dt \\ &= \int (t\mathbf{i} + (t^2 + 1)\mathbf{j} + (2 - \cos(t))\mathbf{k})\,dt \\ &= (t^2/2)\mathbf{i} + (t^3/3 + t)\mathbf{j} + (2t - \sin(t))\mathbf{k} + \mathbf{C}.\end{aligned}$$

Setting $t = 0$ in this equation gives us

$$\begin{aligned}\mathbf{R}_0 &= \mathbf{R}(0) \\ &= \mathbf{0} + \mathbf{C}.\end{aligned}$$

Thus, we have

$$\mathbf{C} = \mathbf{R}_0 = 2\mathbf{i}.$$

This gives us

$$\begin{aligned}\mathbf{R}(t) &= (t^2/2)\mathbf{i} + (t^3/3 + t)\mathbf{j} + (2t - \sin(t))\mathbf{k} + \mathbf{C} \\ &= (t^2/2)\mathbf{i} + (t^3/3 + t)\mathbf{j} + (2t - \sin(t))\mathbf{k} + 2\mathbf{i} \\ &= (t^2/2 + 2)\mathbf{i} + (t^3/3 + t)\mathbf{j} + (2t - \sin(t))\mathbf{k}.\end{aligned}$$

Problem 3.

Alice stands at the edge of a cliff. From 100 feet above the ground she throws a baseball at 88 feet per second, at an angle of 30° above the horizontal. How far from the base of the cliff does the ball land? What is the time of flight? What is the ball's maximum height above the ground? (Neglect air resistance.) Give numerical answers accurate to two decimal places.

Answer:
Use an xy-coordinate system with the origin at the bottom of the cliff. Thus, the initial position of the ball is $\mathbf{R}_0 = (0, 100)$. The initial velocity vector is $\mathbf{V}_0 = 88(\cos(30°)\mathbf{i} + \sin(30°)\mathbf{j}) = 44\sqrt{3}\mathbf{i} + 44\mathbf{j}$.

The force acting on the ball is the force due to gravity, which is $\mathbf{F} = -mg\mathbf{j}$, where m is the mass of the ball. By Newton's law we have $F = m\mathbf{A}$, which gives the equation

$$-mg\jmath = m\mathbf{A},$$

so $\mathbf{A} = -g\mathbf{j}$. Integrating this with respect to t, we get

$$\mathbf{V} = -gt\mathbf{j} + \mathbf{C}.$$

Setting $t = 0$ in this equation shows $\mathbf{C} = \mathbf{V}_0$. Thus,

$$\mathbf{V} = -gt\mathbf{j} + 44\sqrt{3}\mathbf{i} + 44\mathbf{j} = 44\sqrt{3}\mathbf{i} + (-gt + 44)\mathbf{j}.$$

Integrating this equation to get the position vector, we get

$$\mathbf{R} = 44t\sqrt{3} + (-gt^2/2 + 44t)\mathbf{j} + \mathbf{C}$$

Setting $t = 0$ give $\mathbf{C} = \mathbf{R}(0) = 100\mathbf{j}$. Thus, we get

$$\mathbf{R} = 44t\sqrt{3} + (-gt^2/2 + 44t)\mathbf{j} + 100\mathbf{j} = 44t\sqrt{3} + (-gt^2/2 + 44t + 100)\mathbf{j}$$

Thus, the position (x, y) of the ball at time t is

$$\begin{aligned} x &= 44t\sqrt{3} \\ y &= -\frac{1}{2}gt^2 + 44t + 100. \end{aligned}$$

The ball will hit the ground when $y = 0$, so, to find the time of flight, we need to solve the equation

$$0 = -\frac{1}{2}gt^2 + 44t + 10,$$

where $g = 32$. This can be done by the quadratic formula, or by the polynomial solve on the calculator. The answer, to two decimal places, is

$$T \approx 4.23.$$

(seconds).

Setting $t = T$ in the formula for x gives the distance from base of the cliff of the spot where the ball hits the ground. Using T to the full precision of the calculator, we get $x \approx 322.23$ feet.

The highest point on the trajectory occurs when y reaches it's maximum. The time when this occurs can be found by setting $y' = 0$. Thus, the equation to solve is

$$-gt + 44 = 0,$$

which has the solution $t = 44/32 = 11/8$. Plugging this value into the y-equation gives the max hight as

$$y = -\frac{1}{2}g(11/8)^2 + 44(11/8) + 100 = \frac{521}{4} = 130.25$$

feet (exactly).

Problem 4.

A cannon has a muzzle speed of 550 feet per second. A shot is fired from ground level at an angle of 20° and overshoots the target by 50 feet. At what angle should the next shot be fired to hit the target (same muzzle speed)? Neglect air resistance. Give numerical answers accurate to two decimal places.

Answer:
The formula for the range R_f of a shot fired at an angle of α is

$$R_f = \frac{v_0^2}{g} \sin(2\alpha).$$

Let R be the range to the target. The shot lands at $R + 50$, so we have

$$R + 50 = \frac{v_0^2}{g} \sin(2\alpha)$$

where $v_0 = 550$, $g = 32$ and $\alpha = 20°$. Thus,

$$R = -50 + \frac{(550)^2}{32} \sin(40°) \approx 6026.35$$

feet. To find the angle β to hit the target, we need to solve

$$R = \frac{v_0^2}{g} \sin(2\beta)$$

for β. Let $\theta = 2\beta$, so

$$\sin(2\beta) = \sin(\theta) = \frac{gR}{v_0^2}.$$

One solution of this is

$$\theta_1 = \sin^{-1}(gR/v_0^2) \approx 39.60°,$$

which would give $\beta \approx 19.80°$. Presumably, thats the angle we would fire at. There's another solution of course, which would be $\theta = 180 - \theta_1 \approx 140.39$, corresponding to $\beta \approx 70.20°$.

Problem 5.

Consider the curve parametrized in polar coordinates as $r = 5\sin(\theta)$, $\theta = 2t$. Express the velocity and acceleration vectors in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$. Find the speed of the curve.

Answer:
One can, of course, use the formulas on page 655 of the book. Let's do it by taking derivatives. If $\mathbf{R}$ is the position vector, we have $\mathbf{R} = r\mathbf{u}_r = 5\sin(\theta)\mathbf{u}_r$.

Differentiation gives

$$\begin{aligned}
\mathbf{V} &= \frac{d\mathbf{R}}{dt} \\
&= \frac{d}{dt}[5\sin(\theta)\mathbf{u}_r] \\
&= 5\cos(\theta)\frac{d\theta}{dt}\mathbf{u}_r + 5\sin(\theta)\frac{d\mathbf{u}_r}{dt} \\
&= 5\cos(\theta)\frac{d\theta}{dt}\mathbf{u}_r + 5\sin(\theta)\frac{d\theta}{dt}\frac{d\mathbf{u}_r}{d\theta} \\
&= 5\cos(\theta)(2)\mathbf{u}_r + 5\sin(\theta)(2)\mathbf{u}_\theta, \qquad\qquad \frac{d\theta}{dt} = 2, \quad \frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta \\
&= 10\cos(\theta)\mathbf{u}_r + 10\sin(\theta)\mathbf{u}_\theta.
\end{aligned}$$

Differentiating again, we get

$$\begin{aligned}
\mathbf{A} &= \frac{d\mathbf{V}}{dt} \\
&= \frac{d}{dt}[10\cos(\theta)\mathbf{u}_r + 10\sin(\theta)\mathbf{u}_\theta] \\
&= -10\sin(\theta)\frac{d\theta}{dt}\mathbf{u}_r + 10\cos(\theta)\frac{d\mathbf{u}_r}{dt} + 10\cos(\theta)\frac{d\theta}{dt}\mathbf{u}_\theta + 10\sin(\theta)\frac{d\mathbf{u}_\theta}{dt} \\
&= -10\sin(\theta)\frac{d\theta}{dt}\mathbf{u}_r + 10\cos(\theta)\frac{d\theta}{dt}\frac{d\mathbf{u}_r}{d\theta} + 10\cos(\theta)\frac{d\theta}{dt}\mathbf{u}_\theta + 10\sin(\theta)\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{d\theta} \\
&= -10\sin(\theta)(2)\mathbf{u}_r + 10\cos(\theta)(2)\mathbf{u}_\theta + 10\cos(\theta)(2)\mathbf{u}_\theta + 10\sin(\theta)(2)(-\mathbf{u}_r) \\
&= -40\sin(\theta)\mathbf{u}_r + 40\sin(\theta)\mathbf{u}_\theta.
\end{aligned}$$

where we've used

$$\frac{d\theta}{dt} = 2, \qquad \frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta, \qquad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r.$$

To find the speed, use our result that

$$\mathbf{V} = 10\cos(\theta)\mathbf{u}_r + 10\sin(\theta)\mathbf{u}_\theta.$$

This gives us

$$\begin{aligned}
\|\mathbf{V}\|^2 &= \mathbf{V}\cdot\mathbf{V} \\
&= (10\cos(\theta)\mathbf{u}_r + 10\sin(\theta)\mathbf{u}_\theta)\cdot(10\cos(\theta)\mathbf{u}_r + 10\sin(\theta)\mathbf{u}_\theta) \\
&= 100\cos^2(\theta)\mathbf{u}_r\cdot\mathbf{u}_r + 200\cos(\theta)\sin(\theta)\mathbf{u}_r\cdot\mathbf{u}_\theta + 100\sin^2(\theta)\mathbf{u}_\theta\cdot\mathbf{u}_\theta \\
&= 100\cos^2(\theta)(1) + 200\cos(\theta)\sin(\theta)(0) + 100\sin^2(\theta)(1) \\
&= 100\cos^2(\theta) + 100\sin^2(\theta) \\
&= 100,
\end{aligned}$$

since $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are orthogonal unit vectors. The speed is $\|\mathbf{V}\| = \sqrt{100} = 10$.

Problem 6.

Consider the curve in the plane parametrized by $\mathbf{R}(t) = (t, t^2)$. Find the speed, the unit tangent vector, the curvature and the principal unit normal. Find the osculating circle at the point where $t = 1$. Find the scalar tangential and normal components of the acceleration as the moving particle goes through the point where $t = 1$.

Answer:
Differentiate the formula for $\mathbf{R}$ to get

$$\frac{d\mathbf{R}}{dt} = (1, 2t).$$

From this we get

$$\text{speed} = \frac{ds}{dt} = \left\| \frac{d\mathbf{R}}{dt} \right\| = \sqrt{1 + 4t^2}.$$

The unit tangent vector T is given by

$$\mathbf{T} = \frac{\frac{d\mathbf{R}}{dt}}{\|\frac{d\mathbf{R}}{dt}\|} = \frac{1}{\sqrt{1+4t^2}}(1, 2t) = \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t}} \right)$$

Differentiating this gives us

$$\frac{d\mathbf{T}}{dt} = \left(-\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right)$$

Now we have

$$\frac{d\mathbf{T}}{dt} = \frac{ds}{dt}\frac{d\mathbf{T}}{ds},$$

so

$$\begin{aligned}
\frac{d\mathbf{T}}{ds} &= \frac{1}{\frac{ds}{dt}}\frac{d\mathbf{T}}{dt} \\
&= \frac{1}{\sqrt{1+4t^2}} \left(-\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right) \\
&= \left(-\frac{4t}{(1+4t^2)^2}, \frac{2}{(1+4t^2)^2} \right).
\end{aligned}$$

By definition, the curvature κ is given by

$$\begin{aligned}
\kappa &= \left\| \frac{d\mathbf{T}}{ds} \right\| \\
&= \left\{ \left[-\frac{4t}{(1+4t^2)^2} \right]^2 + \left[\frac{2}{(1+4t^2)^2} \right]^2 \right\}^{1/2} \\
&= \frac{2}{(1+4t^2)^{3/2}}
\end{aligned}$$

Finally the principal unit normal $\mathbf{N}$ satisfies

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$$

so we have

$$\begin{aligned}
\mathbf{N} &= \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} \\
&= \frac{(1+4t^2)^{3/2}}{2}\left(-\frac{4t}{(1+4t^2)^2}, \frac{2}{(1+4t^2)^2}\right) \\
&= \left(-\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right)
\end{aligned}$$

Next, lets find the osculating circle at $t = 1$. The position vector $\mathbf{C}$ of the center of the osculating circle is given by

$$\mathbf{C} = \mathbf{R} + \frac{1}{\kappa}\mathbf{N}.$$

At $t = 1$, we have

$$\begin{aligned}
\mathbf{R} &= (1, 1) \\
\kappa &= \frac{2\sqrt{5}}{25} \\
\mathbf{N} &= \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right).
\end{aligned}$$

Plugging into the formula above gives

$$\mathbf{C} = \left(-4, \frac{7}{2}\right).$$

The radius of the osculating circle is

$$\frac{1}{\kappa} = \frac{5\sqrt{5}}{2}$$

The equation of the osculating circle is then

$$(x+4)^2 + (y-7/2)^2 = \frac{125}{4}.$$

The formulas for the tangential compounder of acclamation A_T and the normal component of acclamation A_N are

$$\begin{aligned}
A_T &= \frac{d^2 s}{dt^2} \\
A_N &= \left(\frac{ds}{dt}\right)^2 \kappa
\end{aligned}$$

From $ds/dt = \sqrt{1+4t^2}$, we have

$$\frac{d^2s}{dt^2} = \frac{4t}{\sqrt{1+4t^2}}$$

Plugging $t = 1$, we find

$$A_T = \frac{4\sqrt{5}}{5}$$

at $t = 1$. At $t = 1$, $ds/dt = \sqrt{5}$, and we have the value of κ at $t = 1$ above, so

$$A_N = \frac{2\sqrt{5}}{5}$$

Problem 7.

A stunt driver wants to drive a motorcycle around a vertical loop with a radius of 50 feet. How fast must the motorcycle be traveling at the top of the loop so it doesn't fall off?

Answer:
The centripetal force will be

$$F = m\frac{v_0^2}{r}$$

where r is the radius of the circle, v_0 is the speed, and m is the mass of the motorcycle and driver. At the top of the loop, the centripetal force must at least balance the force due to gravity, which is

$$F_{\mathrm{grav}} = mg.$$

Thus, the slowest speed that will work is the one that satisfies the equation

$$m\frac{v_0^2}{r} = mg.$$

Solving this for v_0 gives

$$v_0 = \sqrt{rg}.$$

In the present case we have $r = 50$ and $g = 32$, which gives

$$v_0 = 40$$

feet per second. That's about 27.27 miles per hour.

Problem 8.

A satellite follows a stable circular orbit 100 miles above the surface of the earth. Find the orbital velocity and period of the satellite.

Use the following values: $g = 32\,\mathrm{ft}/\mathrm{sec}^2$, the radius of the earth is $R_e = 3963\,\mathrm{mi}$, and $GM = gR_e^2$.

Answer:
The acceleration necessary to keep the satallie in the circlular orbit is

$$\frac{v_0^2}{r},$$

where v_0 is the orbital speed and r is the radius of the orbit, which has its center at the center of the earth. Thus, the force required is

$$F_{\mathrm{orbit}} = m\frac{v_0^2}{r},$$

where m is the mass of the satellite. This force must be provided by the force of gravity, which is

$$F_{\mathrm{grav}} = \frac{GMm}{r^2},$$

where M is the mass of the earth and G is Newton's constant. The two forces must be the same, so we have

$$m\frac{v_0^2}{r} = \frac{GMm}{r^2},$$

Solving this for v_0 gives

$$v_0 = \sqrt{\frac{GM}{r}}.$$

Using the information that $GM = gR_e^2$, we can rewite this as

$$v_0 = \sqrt{\frac{gR_e^2}{r}}.$$

We can now plug in some values, but we have to decide which units to use, say feet and seconds. We have

$$R_e = 3963\,\mathrm{mi} = 3963(5280)\,\mathrm{ft} = 20,924,640\,\mathrm{ft}.$$

If R is the distance of the satellite from the surface of the earth, we have

$$R = 100\,\mathrm{mi} = 5280,000\,\mathrm{ft}.$$

Then the radius r of the orbit is

$$r = R + R_e = 21,452,640\,\mathrm{ft}.$$

Plugging these figures into the formulas gives

$$v_0 \approx 25,555.98\,\mathrm{ft}/\mathrm{sec},$$

which is about $17,425$ miles per hour.

The distance that the satellite travels to go once around the orbit is the circufrance of the orbit $2\pi r$, and the speed is v_0, so the time T to complete one orbit is

$$T = \frac{2\pi r}{v_0}.$$

Plugging into this formula gives

$$T \approx 5274.34 \, \text{sec},$$

or about 1 hour, 27 minutes and 54 seconds.
