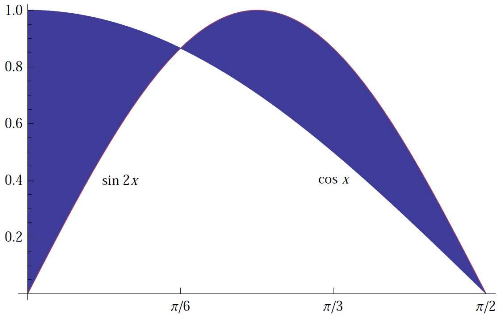
Calculus II - Fall 2013

Midterm Exam II, November 12, 2013

In the following problems you are required to show all your work and provide the necessary explanations everywhere to get full credit.

1. Find the area between the graphs of $y = \cos x$ and $y = \sin 2x$ for $0 \le x \le \pi/2$.



Solution: Note that

$$\cos x = \sin 2x \quad \text{if} \quad x = \frac{\pi}{6},\ \frac{\pi}{2}$$

Therefore

$$\begin{aligned}
A &= \int_0^{\pi/6} (\cos x - \sin 2x)dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)dx \\
&= \left[\sin x + \frac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2} \\
&= \left(\sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} - \sin 0 - \frac{1}{2}\cos 0\right) + \left(-\frac{1}{2}\cos\pi - \sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{3} + \sin\frac{\pi}{6}\right) \\
&= \left(\frac{1}{2} + \frac{1}{4} - 0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{2}\right) = \frac{1}{2}
\end{aligned}$$

2. Find the volume of the solid obtaining by rotating the region bounded by $y = x^2$, $x = 1$, and $y = 0$

Solution:

(a) We have

$$V = \int_0^1 \pi (x^2)^2 dx = \pi \int_0^1 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^1 = \frac{\pi}{5}$$

(b) We have

$$V = \int_0^1 2\pi (1-x) x^2 dx = 2\pi \int_0^1 (x^2 - x^3) dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$

or

$$V = \int_0^1 \pi (1 - \sqrt{y})^2 dy = \pi \int_0^1 (1 - 2\sqrt{y} + y) dy = \pi \left[y - \frac{4}{3} y^{3/2} + \frac{y^2}{2} \right]_0^1 = \frac{\pi}{6}$$

(c) We have

$$V = \int_0^1 2\pi x \cdot x^2 dx = 2\pi \int_0^1 x^3 dx = 2\pi \left[\frac{x^4}{4} \right]_0^1 = \frac{\pi}{2}$$

or

$$V = \int_0^1 \pi (\sqrt{y})^2 dy = \pi \int_0^1 y dy = \pi \left[\frac{y^2}{2} \right]_0^1 = \frac{\pi}{2}$$

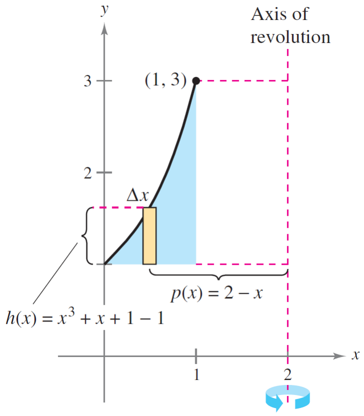
(d) We have

$$V = \int_0^1 \pi [(1 + x^2)^2 - 1^2] dx = \pi \int_0^1 (2x^2 + x^4) dx = \pi \left[\frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1 = \frac{13\pi}{15}$$

or

$$V = \int_0^1 2\pi (y+1)(1-\sqrt{y}) dy = 2\pi \int_0^1 (y - y^{3/2} + 1 - y^{1/2}) dy = 2\pi \left[\frac{y^2}{2} - \frac{2y^{5/2}}{5} + y - \frac{2y^{3/2}}{3} \right]_0^1 = \frac{13\pi}{15}$$

3. Find the volume of the solid obtaining by rotating the region bounded by $y = x^3 + x + 1,\ x = 1$, and $y = 1$ about the line $x = 2$.



Solution: We have

$$2\pi(\text{radius})(\text{height}) = 2\pi \cdot (2 - x) \cdot (x^3 + x + 1 - 1) = 2\pi(2 - x)(x^3 + x) = 2\pi(-x^4 + 2x^3 - x^2 + 2x)$$

Therefore

$$V = \int_0^1 2\pi \cdot (2 - x) \cdot (x^3 + x + 1 - 1)dx$$

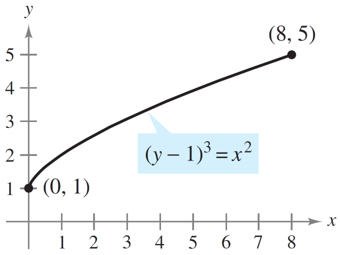
$$= 2\pi \int_0^1 (-x^4 + 2x^3 - x^2 + 2x)dx$$

$$= 2\pi \left[-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2 \right]_0^1$$

$$= 2\pi \left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1 \right)$$

$$= \frac{29\pi}{15}$$

4. Find the arc length of the graph of $(y-1)^3 = x^2$ on the interval $[0,8]$ as shown in the Figure below.



Solution: We have

$$(y-1)^3 = x^2 \quad \Longrightarrow \quad x = (y-1)^{3/2} \quad \Longrightarrow \quad \frac{dx}{dy} = \frac{3}{2}(y-1)^{1/2}$$

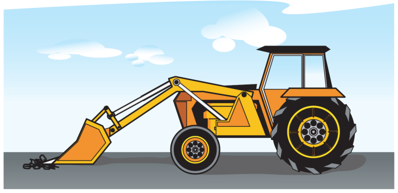
and so the arc length formula gives

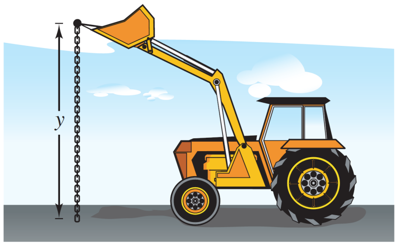
$$L = \int_1^5 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_1^5 \sqrt{1 + \left[\frac{3}{2}(y-1)^{1/2}\right]^2} dy = \int_1^5 \sqrt{1 + \frac{9}{4}(y-1)dy}$$

$$= \int_1^5 \sqrt{1 + \frac{9}{4}y - \frac{9}{4}} dy = \int_1^5 \sqrt{\frac{9}{4}y - \frac{5}{4}} dy = \left[\begin{array}{l} \frac{9}{4}y - \frac{5}{4} = u \\ d\left(\frac{9}{4}y - \frac{5}{4}\right) = du \\ \frac{9}{4}dy = du \\ dy = \frac{4}{9}dv \end{array}\right] = \frac{4}{9}\int_{\frac{9}{4}\cdot 1 - \frac{5}{4}}^{\frac{9}{4}\cdot 5 - \frac{5}{4}} \sqrt{u} du$$

$$= \frac{4}{9}\int_1^{10} \sqrt{u} du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2}\Big]_1^{10} = \frac{8}{27}\left(10^{3/2} - 1^{3/2}\right) = \frac{8}{27}\left(10\sqrt{10} - 1\right) \approx 9.073$$

5. A 20-foot chain weighing 5 pounds per foot is lying coiled on the ground. How much work is required to raise one end of the chain to a height of 20 feet so that it is fully extended?





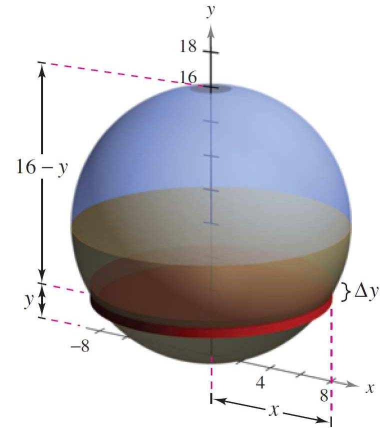
Solution: Let's place the origin at the top of the fully extended chain and the x-axis pointing downward. We divide the chain into small parts with length Δx. If x_i^* is a point in the ith such interval, then all points in the interval are lifted by approximately the same amount, namely x_i^*. The chain weighs 5 pounds per foot, so the weight of the ith part is $5\Delta x$. Thus the work done on the ith part, in foot-pounds, is

$$\underbrace{(5\Delta x)}_{\text{force}} \cdot \underbrace{x_i^*}_{\text{distance}} = 5x_i^*\Delta x$$

We get the total work done by adding all these approximations and letting the number of parts become large (so $\Delta x \to 0$):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 5x_i^*\Delta x = \int_0^{20} 5x\,dx = 5\int_0^{20} x\,dx = \left[5 \cdot \frac{x^2}{2}\right]_0^{20} = 5 \cdot \frac{20^2}{2} = 1000 \text{ ft-lb}$$

6. A spherical tank of radius 8 feet is half full of oil that weighs 50 pounds per cubic foot. Find the work required to pump oil out through a hole in the top of the tank.



Solution: Let's measure depths from the bottom of the tank by introducing a vertical coordinate line. The oil extends from a depth of 8 ft to a depth of 16 ft and so we divide the interval $[8, 16]$ into n subintervals with endpoints $y_0, y_1, \ldots, y_n$ and choose y_i^* in the ith subinterval. This divides the oil into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δy. For a circle of radius r_i and center at y_i^* we have

$$(8 - y_i^*)^2 + r_i^2 = 8^2 \quad \Longrightarrow \quad r_i^2 = 64 - (8 - y_i^*)^2 = 64 - 64 + 16y_i^* - (y_i^*)^2 = 16y_i^* - (y_i^*)^2$$

Thus an approximation to the volume of the ith layer of water is

$$V_i \approx \pi r_i^2 \Delta y = \pi(16y_i^* - (y_i^*)^2)\Delta y$$

and so the force required to raise this layer is

$$F_i = \text{density} \times \text{volume} \approx 50\pi(16y_i^* - (y_i^*)^2)\Delta y$$

Each particle in the layer must travel a distance of approximately $16 - y_i^*$. The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance $16 - y_i^*$:

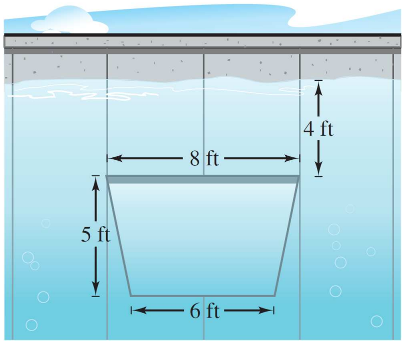
$$W_i \approx F_i x_i^* \approx 50\pi(16 - y_i^*)(16y_i^* - (y_i^*)^2)\Delta y$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 50\pi(16 - y_i^*)(16y_i^* - (y_i^*)^2)\Delta y = \int_{0}^{8} 50\pi(16 - y)(16y - y^2)dy \\ &= 50\pi \int_{0}^{8} \left(256y - 32y^2 + y^3\right) dy = 50\pi \left[128y^2 - \frac{32}{3}y^3 + \frac{1}{4}y^4\right]_0^8 \\ &= 50\pi \cdot \frac{11,264}{3} \approx 589,782 \text{ ft-lb} \end{aligned}$$

Other answers: $W = \int_{0}^{8} 50\pi(8 + y)(64 - y^2)dy = \int_{-8}^{0} 50\pi(8 - y)(64 - y^2)dy.$

7. A vertical gate in a dam has the shape of an isosceles trapezoid 8 feet across the top and 6 feet across the bottom, with a height of 5 feet, as shown in the Figure below. What is the fluid force on the gate when the top of the gate is 4 feet below the surface of the water?



Solution: We choose a vertical x-axis with origin at the surface of the water. We divide the interval $[4, 9]$ into subintervals of equal length with endpoints x_i and we choose $x_i^* \in [x_{i-1}, x_i]$. The ith horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i, where, from similar triangles,

$$\frac{a}{9-x_i^*} = \frac{1}{5} \quad \Longrightarrow \quad a = \frac{1}{5}(9-x_i^*)$$

and so

$$w_i = 2(3+a) = 2\left(3+\frac{1}{5}(9-x_i^*)\right) = 2\left(3+\frac{9}{5}-\frac{1}{5}x_i^*\right) = 2\left(\frac{24}{5}-\frac{1}{5}x_i^*\right) = \frac{2}{5}\left(24-x_i^*\right)$$

If A_i is the area of the ith strip, then

$$A_i \approx w_i \Delta x = \frac{2}{5}\left(24-x_i^*\right)\Delta x$$

If Δx is small, then the pressure P_i on the ith strip is almost constant, therefore

$$P_i \approx \delta x_i^* = 62.5 x_i^*$$

The hydrostatic force F_i acting on the ith strip is the product of the pressure and the area:

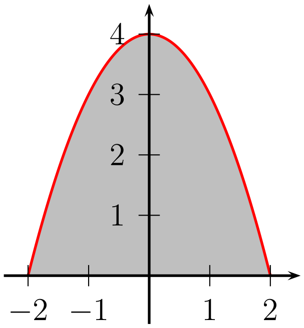
$$F_i = P_i A_i \approx 62.5 x_i^* \cdot \frac{2}{5}\left(24-x_i^*\right)\Delta x = 25 x_i^*\left(24-x_i^*\right)\Delta x$$

Adding these forces and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam:

$$\begin{aligned} F &= \lim_{n\to\infty}\sum_{i=1}^{n} 25x_i^*\left(24-x_i^*\right)\Delta x = \int_4^9 25x\left(24-x\right)dx \\ &= 25\int_4^9 \left(24x-x^2\right)dx = 25\left[12x^2-\frac{x^3}{3}\right]_4^9 = \frac{41875}{3} \approx 13958.3 \text{ lb} \end{aligned}$$

Other answers: $F = 62.5\displaystyle\int_0^5 \left(6+\frac{2}{5}x\right)(9-x)dx = 62.5\int_0^5\left(8-\frac{2}{5}x\right)(4+x)dx.$

8. Find the center of mass of the lamina of uniform density ρ bounded by the graph of $f(x) = 4 - x^2$ and the x-axis.



Solution: There is no need to use the formula to calculate $\overline{x}$ because, by the symmetry principle, the center of mass must lie on the y-axis, so $\overline{x} = 0$. To find $\overline{y}$, we use the formula

$$\overline{y} = \frac{\displaystyle\int_a^b \frac{1}{2}[f(x)]^2 dx}{\displaystyle\int_a^b f(x) dx}$$

We have

$$\int_{-2}^{2} \frac{1}{2}(4 - x^2)^2 dx = \frac{1}{2}\int_{-2}^{2}(16 - 8x^2 + x^4)dx = \frac{1}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{256}{15}$$

and

$$\int_{-2}^{2}(4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$

therefore

$$\overline{y} = \frac{256/15}{32/3} = \frac{8}{5}$$

The center of mass is located at the point $\left(0, \dfrac{8}{5}\right)$.

9. Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xydx + e^{-x^2}(y^2 - 1)dy = 0$$

Solution: Note that $y = 0$ is a solution of the differential equation – but this solution does not satisfy the initial condition. So, we can assume that $y \neq 0$. To separate variables, we must rid the first term of y and the second term of e^{-x^2}. We have

$$xydx + e^{-x^2}(y^2 - 1)dy = 0$$

$$e^{-x^2}(y^2 - 1)dy = -xydx$$

$$\frac{y^2 - 1}{y}dy = -xe^{x^2}dx$$

$$\int \frac{y^2 - 1}{y}dy = -\int xe^{x^2}dx$$

Since

$$\int \frac{y^2 - 1}{y}dy = \int \left(\frac{y^2}{y} - \frac{1}{y}\right)dy = \int \left(y - \frac{1}{y}\right)dy = \frac{y^2}{2} - \ln|y| + C_1$$

and

$$\int xe^{x^2}dx = \begin{bmatrix} x^2 = u \\ d\left(x^2\right) = du \\ 2xdx = du \\ xdx = \dfrac{1}{2}du \end{bmatrix} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C_2 = \frac{1}{2}e^{x^2} + C_2$$

it follows that

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + C$$

From the initial condition $y(0) = 1$, we have

$$\frac{1}{2} - 0 = -\frac{1}{2} + C$$

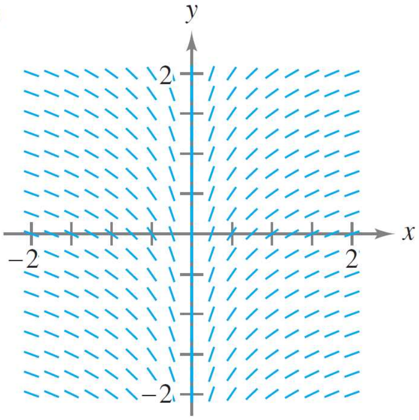
which implies that $C = 1$. So, the particular solution has the implicit form

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + 1$$

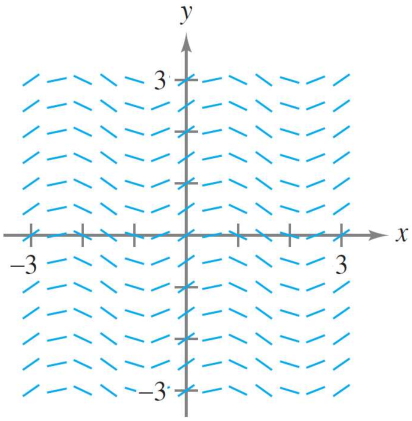
10. Match the differential equation with its slope field. **Explain!**

(I) $\dfrac{dy}{dx} = \cos 2x$ (II) $\dfrac{dy}{dx} = \dfrac{1}{2}\sin x$ (III) $\dfrac{dy}{dx} = e^{-2x}$ (IV) $\dfrac{dy}{dx} = \dfrac{1}{x}$

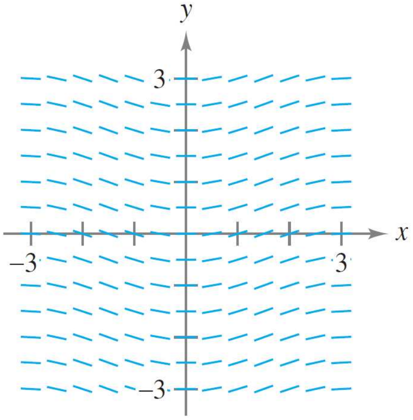
(a)



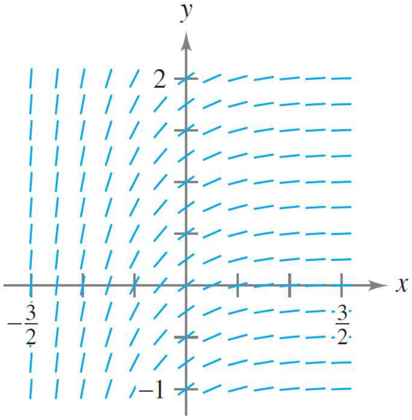
(b)



(c)



(d)



Answer: (I) → (b), (II) → (c), (III) → (d), (IV) → (a).

11. Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{1}{7}, \quad \frac{3}{9}, \quad \frac{7}{11}, \quad \frac{15}{13}, \quad \frac{31}{15}, \ \ldots$$

Then determine whether the particular sequence you have chosen converges or diverges.

Solution: We have

$$a_n = \frac{2^n - 1}{2n + 5}$$

The sequence diverges, since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n - 1}{2n + 5} = \left[\frac{\infty}{\infty}\right] = \lim_{n\to\infty} \frac{(2^n - 1)'}{(2n + 5)'} = \lim_{n\to\infty} \frac{2^n \ln 2}{2} = \infty$$

12. Prove that the sequence $a_n = \dfrac{2n}{1+n}$ is monotonic.

Solution 1: This sequence is monotonic because each successive term is larger than its predecessor. Indeed, we have

$$a_n = \frac{2n}{1+n} \overset{?}{<} \frac{2(n+1)}{1+(n+1)} = a_{n+1}$$

$$\frac{2n}{1+n} \overset{?}{<} \frac{2n+2}{n+2}$$

$$2n(n+2) \overset{?}{<} (2n+2)(1+n)$$

$$2n^2 + 4n \overset{?}{<} 2n + 2n^2 + 2 + 2n$$

$$2n^2 + 4n \overset{?}{<} 2n^2 + 4n + 2$$

$$0 < 2$$

Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid.

Solution 2: This sequence is monotonic because $f(x) = \dfrac{2x}{1+x}$ is an increasing function. Indeed, we have

$$f'(x) = \left(\frac{2x}{1+x}\right)' = \frac{(2x)'(1+x) - 2x(1+x)'}{(1+x)^2} = \frac{2(1+x) - 2x \cdot 1}{(1+x)^2} = \frac{2 + 2x - 2x}{(1+x)^2} = \frac{2}{(1+x)^2}$$

Since $f'(x) = \dfrac{2}{(1+x)^2} > 0$, the function f is increasing.