Calculus II - Fall 2013

Midterm Exam I, October 3, 2013

MC (10 points). *This part consists of 5 multiple choice problems. Nothing more than the answer is required; consequently no partial credit will be awarded.*

1. Find $\displaystyle\int \frac{dx}{2x+7}$.

Ⓐ $\ln|2x+7| + C$

Ⓑ $2\ln|2x+7| + C$

Ⓒ $\dfrac{1}{2}\ln|2x+7| + C$ ⟵ Correct Answer

Ⓓ $\dfrac{1}{7}\ln|2x+7| + C$

Ⓔ $\dfrac{7}{2}\ln|2x+7| + C$

Solution:

$$\int \frac{dx}{2x+7} = \begin{bmatrix} 2x+7 = u \\ d\,(2x+7) = du \\ 2dx = du \\ dx = \frac{1}{2}du \end{bmatrix} = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x+7| + C$$

2. Find $\displaystyle\int \sqrt{3x+5}\,dx$.

Ⓐ $\dfrac{2}{9}(3x+5)^{3/2} + C$ ⟵ Correct Answer

Ⓑ $\dfrac{1}{3}(3x+5)^{3/2} + C$

Ⓒ $\dfrac{1}{5}(3x+5)^{3/2} + C$

Ⓓ $3(3x+5)^{-1/2} + C$

Ⓔ $3(3x+5)^{3/2} + C$

Solution:

$$\int \sqrt{3x+5}\,dx = \int (3x+5)^{1/2}dx = \begin{bmatrix} 3x+5 = u \\ d\,(3x+5) = du \\ 3dx = du \\ dx = \frac{1}{3}du \end{bmatrix} = \frac{1}{3}\int u^{1/2}du = \frac{1}{3}\cdot\frac{u^{1/2+1}}{1/2+1} + C$$

$$= \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{(3x+5)^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}(3x+5)^{3/2} + C = \frac{2}{9}(3x+5)^{3/2} + C$$

3. Find $\int \sin(1-5x)dx$.

Ⓐ $-\frac{1}{5}\cos(5x-1)+C$

Ⓑ $\frac{1}{5}\cos(5x-1)+C$ ⟵ Correct Answer

Ⓒ $5\cos(5x-1)+C$

Ⓓ $-5\cos(5x-1)+C$

Ⓔ None of the above

Solution:

$$\int \sin(1-5x)dx = \begin{bmatrix} 1-5x=u \\ d(1-5x)=du \\ -5dx=du \\ dx=-\frac{1}{5}du \end{bmatrix} = -\frac{1}{5}\int \sin u du = -\frac{1}{5}(-\cos u)+C = \frac{1}{5}\cos u + C$$

$$= \frac{1}{5}\cos(1-5x)+C = \frac{1}{5}\cos(5x-1)+C$$

4. Find $\int e^{(2+x)/3}dx$.

Ⓐ $e^{(2+x)/3}+C$

Ⓑ $2e^{(2+x)/3}+C$

Ⓒ $\frac{1}{2}e^{(2+x)/3}+C$

Ⓓ $\frac{1}{3}e^{(2+x)/3}+C$

Ⓔ $3e^{(2+x)/3}+C$ ⟵ Correct Answer

Solution:

$$\int e^{(2+x)/3}dx = \int e^{\frac{2+x}{3}}dx = \int e^{\frac{2}{3}+\frac{1}{3}x}dx = \begin{bmatrix} \frac{2}{3}+\frac{1}{3}x=u \\ d\left(\frac{2}{3}+\frac{1}{3}x\right)=du \\ \frac{1}{3}dx=du \\ dx=3du \end{bmatrix} = 3\int e^u du = 3e^u+C = 3e^{(2+x)/3}+C$$

5. Find $\int \frac{dx}{1+x^2}$.

Ⓐ $\ln(1+x^2)+C$

Ⓑ $\arcsin x + C$

Ⓒ $\arccos x + C$

Ⓓ $\arctan x + C$ ⟵ Correct Answer

Ⓔ $\tan x + C$

FR (40 points). *This part consists of 10 questions. Each question will be graded on a pass/fail basis. You are required to show all your work and provide the necessary explanations everywhere to get full credit.*

Find the following integrals:

1. $\displaystyle\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}dx$

Solution:

$$\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}dx = \int \frac{x^{2/3}-x^{1/4}}{x^{1/2}}dx = \int \left(\frac{x^{2/3}}{x^{1/2}} - \frac{x^{1/4}}{x^{1/2}}\right)dx = \int \left(x^{2/3-1/2} - x^{1/4-1/2}\right)dx$$

$$= \int \left(x^{1/6} - x^{-1/4}\right)dx = \frac{x^{1/6+1}}{1/6+1} - \frac{x^{-1/4+1}}{-1/4+1} = \frac{x^{7/6}}{7/6} - \frac{x^{3/4}}{3/4} + C = \boxed{\frac{6}{7}x^{7/6} - \frac{4}{3}x^{3/4} + C}$$

2. $\displaystyle\int_0^{\pi/6} (1+\sin x)^5 \cos x dx$

Solution 1:

$$\int_0^{\pi/6} (1+\sin x)^5 \cos x dx = \begin{bmatrix} 1+\sin x = u \\ d(1+\sin x) = du \\ \cos x dx = du \end{bmatrix} = \int_{1+\sin 0}^{1+\sin(\pi/6)} u^5 du = \int_1^{3/2} u^5 du = \frac{u^6}{6}\bigg]_1^{3/2}$$

$$= \frac{(3/2)^6}{6} - \frac{1^6}{6} = \frac{3^6}{6\cdot 2^6} - \frac{1}{6} = \frac{3^6}{6\cdot 2^6} - \frac{2^6}{6\cdot 2^6} = \frac{3^6-2^6}{6\cdot 2^6} = \frac{729-64}{6\cdot 64} = \boxed{\frac{665}{384}}$$

Solution 2: We have

$$\int (1+\sin x)^5 \cos x dx = \begin{bmatrix} 1+\sin x = u \\ d(1+\sin x) = du \\ \cos x dx = du \end{bmatrix} = \int u^5 du = \frac{u^6}{6} + C = \frac{(1+\sin x)^6}{6} + C$$

Therefore

$$\int_0^{\pi/6} (1+\sin x)^5 \cos x dx = \frac{(1+\sin x)^6}{6}\bigg]_0^{\pi/6} = \frac{(1+\sin(\pi/6))^6}{6} - \frac{(1+\sin 0)^6}{6} = \frac{(1+1/2)^6}{6} - \frac{(1+0)^6}{6}$$

$$= \frac{(3/2)^6}{6} - \frac{1^6}{6} = \frac{3^6}{6\cdot 2^6} - \frac{1}{6} = \frac{3^6}{6\cdot 2^6} - \frac{2^6}{6\cdot 2^6} = \frac{3^6-2^6}{6\cdot 2^6} = \frac{729-64}{6\cdot 64} = \boxed{\frac{665}{384}}$$

3. $\displaystyle\int x^5 \ln 7x dx$

Solution:

$$\int x^5 \ln 7x dx \left[\begin{array}{c|c} \ln 7x = u & x^5 dx = dv \\ d(\ln 7x) = du & \dfrac{x^6}{6} = v \\ \dfrac{1}{x} dx = du & \end{array}\right] = \ln 7x \cdot \frac{x^6}{6} - \int \frac{x^6}{6} \cdot \frac{1}{x} dx = \frac{1}{6} x^6 \ln 7x - \frac{1}{6} \int x^5 dx$$

$$= \frac{1}{6} x^6 \ln 7x - \frac{1}{6} \cdot \frac{x^6}{6} + C$$

$$= \boxed{\frac{1}{6} x^6 \ln 7x - \frac{1}{36} x^6 + C}$$

4. $\displaystyle\int \cos^2 7x dx$

Solution: We have

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \quad \Longrightarrow \quad \cos^2 7x = \frac{1 + \cos 14x}{2}$$

Therefore

$$\int \cos^2 7x dx = \int \left(\frac{1 + \cos 14x}{2}\right) dx = \int \left(\frac{1}{2} + \frac{\cos 14x}{2}\right) dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 14x dx$$

$$= \frac{1}{2} x + \frac{1}{2} \int \cos 14x dx = \left[\begin{array}{l} 14x = u \\ d\,(14x) = du \\ 14 dx = du \\ dx = \dfrac{1}{14} du \end{array}\right] = \frac{1}{2} x + \frac{1}{2} \cdot \frac{1}{14} \int \cos u du = \frac{1}{2} x + \frac{1}{28} \sin u + C$$

$$= \boxed{\frac{1}{2} x + \frac{1}{28} \sin 14x + C}$$

5. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-2}}dx$

Solution 1:
METHOD OF INTEGRATION:
(i) If $\sqrt{a^2-x^2}$, then $x = a\sin\theta,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.
(ii) If $\sqrt{a^2+x^2}$, then $x = a\tan\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.
(iii) If $\sqrt{x^2-a^2}$, then $x = a\sec\theta,\ 0 \le \theta < \dfrac{\pi}{2}$ or $\pi \le \theta < \dfrac{3\pi}{2}$.
We have

$$\int \frac{1}{x^2\sqrt{x^2-2}}dx = \left[\begin{array}{c} x=\sqrt{2}\sec\theta \\ dx = d(\sqrt{2}\sec\theta) \\ dx = \sqrt{2}\sec\theta\tan\theta d\theta \\ \sqrt{x^2-2} = \sqrt{2\sec^2\theta-2} = \sqrt{2(\sec^2\theta-1)} = \sqrt{2\tan^2\theta} = \sqrt{2}|\tan\theta| = \sqrt{2}\tan\theta \end{array}\right]$$

$$= \int \frac{1}{2\sec^2\theta\cdot\sqrt{2}\tan\theta}\cdot\sqrt{2}\sec\theta\tan\theta d\theta = \frac{1}{2}\int\cos\theta d\theta = \frac{1}{2}\sin\theta + C$$

Note that $\sin\theta = \dfrac{\sqrt{x^2-2}}{x}$, therefore $\displaystyle\int \frac{1}{x^2\sqrt{x^2-2}}dx = \boxed{\frac{1}{2}\cdot\frac{\sqrt{x^2-2}}{x}+C}$.

Solution 2: We first note that

$$\int \frac{1}{x^2\sqrt{x^2-2}}dx = \int \frac{\sqrt{x^2-2}}{x^2(x^2-2)}dx = \frac{1}{2}\int\left(\frac{\sqrt{x^2-2}}{x^2-2} - \frac{\sqrt{x^2-2}}{x^2}\right)dx$$

$$= \frac{1}{2}\int\left(\frac{1}{\sqrt{x^2-2}} - \frac{\sqrt{x^2-2}}{x^2}\right)dx = \frac{1}{2}\int\frac{1}{\sqrt{x^2-2}}dx - \frac{1}{2}\int\frac{\sqrt{x^2-2}}{x^2}dx$$

We have

$$\int\frac{\sqrt{x^2-2}}{x^2}dx = \int\sqrt{x^2-2}\cdot\frac{1}{x^2}dx = \left[\begin{array}{c|c} \sqrt{x^2-2}=u & \frac{1}{x^2}dx = dv \\ d(\sqrt{x^2-2}) = du & -\frac{1}{x} = v \\ \frac{x}{\sqrt{x^2-2}}dx = du & \end{array}\right]$$

$$= \sqrt{x^2-2}\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{x}{\sqrt{x^2-2}}dx = -\frac{\sqrt{x^2-2}}{x} + \int\frac{1}{\sqrt{x^2-2}}dx$$

Therefore

$$\int \frac{1}{x^2\sqrt{x^2-2}}dx = \frac{1}{2}\int\frac{1}{\sqrt{x^2-2}}dx - \frac{1}{2}\int\frac{\sqrt{x^2-2}}{x^2}dx$$

$$= \frac{1}{2}\int\frac{1}{\sqrt{x^2-2}}dx + \frac{1}{2}\cdot\frac{\sqrt{x^2-2}}{x} - \frac{1}{2}\int\frac{1}{\sqrt{x^2-2}}dx = \frac{1}{2}\cdot\frac{\sqrt{x^2-2}}{x} + C$$

Solution 3 (version 1): Note that $\int \frac{1}{x^2\sqrt{x^2-2}}dx$ can be rewritten as $\int x^m(a+bx^n)^p dx$ with

$$m=-2,\ n=2,\ p=-\frac{1}{2},\ a=-2,\ \text{and } b=1$$

It is known that if

$$\frac{m+1}{n}+p \text{ is an integer (which is exactly the case)}$$

then

$$b+\frac{a}{x^n}=u^N,\ \text{where } N \text{ is the denominator of } p$$

Therefore we have

$$\int \frac{1}{x^2\sqrt{x^2-2}}dx \;=\; \left[\begin{array}{l} 1-\frac{2}{x^2}=u^2 \implies \sqrt{1-\frac{2}{x^2}}=u \implies \frac{\sqrt{x^2-2}}{x}=u \\ d\left(1-\frac{2}{x^2}\right)=du^2 \\ \frac{4}{x^3}dx=2udu \\ \frac{1}{x^3}\cdot\frac{1}{u}dx=\frac{1}{2}du \implies \frac{1}{x^3}\cdot\frac{x}{\sqrt{x^2-2}}dx=\frac{1}{2}du \implies \frac{1}{x^2\sqrt{x^2-2}}dx=\frac{1}{2}du \end{array}\right]$$

$$=\frac{1}{2}\int du=\frac{1}{2}u+C=\frac{1}{2}\cdot\frac{\sqrt{x^2-2}}{x}+C$$

Solution 3 (version 2): We have

$$\int \frac{1}{x^2\sqrt{x^2-2}}dx=\left[\begin{array}{l} \frac{\sqrt{x^2-2}}{x}=u \\ d\left(\frac{\sqrt{x^2-2}}{x}\right)=du \\ \frac{2}{x^2\sqrt{x^2-2}}dx=du \\ \frac{1}{x^2\sqrt{x^2-2}}dx=\frac{1}{2}du \end{array}\right]=\frac{1}{2}\int du=\frac{1}{2}u+C=\frac{1}{2}\cdot\frac{\sqrt{x^2-2}}{x}+C$$

6. $\int \frac{4x+1}{x^2-4}dx$

Solution: Since

$$x^2-4 = x^2-2^2 = (x-2)(x+2)$$

this is a Type I integral. Therefore first we find constants A and B such that

$$\frac{4x+1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

We have

$$\frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

therefore

$$4x+1 = A(x+2)+B(x-2) \tag{1}$$

We now can proceed in two different ways:

Method 1: If we expand the parentheses on the right-hand side of (1) and collect like terms, we get

$$4x+1 = (A+B)x + (2A-2B)$$

hence

$$\begin{cases} A+B=4 \\ 2A-2B=1 \end{cases} \implies A=\frac{9}{4} \quad \text{and} \quad B=\frac{7}{4}$$

Method 2: If we put $x=-2$ in (1), we get

$$4(-2)+1 = B\cdot(-4) \implies B=\frac{7}{4}$$

Similarly, if we put $x=2$ in (1), we get

$$4(2)+1 = A\cdot 4 \implies A=\frac{9}{4}$$

So, we have

$$\frac{4x+1}{x^2-4} = \frac{9}{4}\cdot\frac{1}{x-2} + \frac{7}{4}\cdot\frac{1}{x+2}$$

therefore

$$\int \frac{4x+1}{x^2-4}dx = \frac{9}{4}\int\frac{dx}{x-2} + \frac{7}{4}\int\frac{dx}{x+2} = \boxed{\frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + C}$$

7. $\int\limits_1^\infty \frac{1}{(2x+1)^2}dx$

Solution 1: We have

$$\int \frac{1}{(2x+1)^2}dx = \begin{bmatrix} 2x+1=u \\ d\,(2x+1)=du \\ 2dx=du \\ dx=\frac{1}{2}du \end{bmatrix} = \frac{1}{2}\int \frac{1}{u^2}du = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\cdot\frac{u^{-2+1}}{-2+1}+C = \frac{1}{2}\cdot\frac{u^{-1}}{-1}+C$$

$$= -\frac{1}{2u}+C$$

$$= -\frac{1}{2(2x+1)}+C$$

Therefore

$$\int\limits_1^\infty \frac{1}{(2x+1)^2}dx = \lim_{t\to\infty}\int\limits_1^t \frac{1}{(2x+1)^2}dx = \lim_{t\to\infty}\left[-\frac{1}{2(2x+1)}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{2(2t+1)}+\frac{1}{2(2\cdot1+1)}\right) = \boxed{\frac{1}{6}}$$

Solution 2: We have

$$\int\limits_1^\infty \frac{1}{(2x+1)^2}dx = \lim_{t\to\infty}\int\limits_1^t \frac{1}{(2x+1)^2}dx = \begin{bmatrix} 2x+1=u \\ d\,(2x+1)=du \\ 2dx=du \\ dx=\frac{1}{2}du \end{bmatrix} = \lim_{t\to\infty}\frac{1}{2}\int\limits_{2\cdot1+1}^{2t+1}\frac{1}{u^2}du = \lim_{t\to\infty}\frac{1}{2}\int\limits_3^{2t+1}u^{-2}du$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\frac{u^{-2+1}}{-2+1}\right]_3^{2t+1} = \frac{1}{2}\lim_{t\to\infty}\left[\frac{u^{-1}}{-1}\right]_3^{2t+1} = \frac{1}{2}\lim_{t\to\infty}\left[-\frac{1}{u}\right]_3^{2t+1} = \frac{1}{2}\lim_{t\to\infty}\left(-\frac{1}{2t+1}+\frac{1}{3}\right) = \boxed{\frac{1}{6}}$$

8. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{1-x}}dx$

Solution: We have

$$\int \frac{1}{\sqrt[3]{1-x}}dx = \int (1-x)^{-1/3}dx = \begin{bmatrix} 1-x=u \\ d\,(1-x)=du \\ -dx=du \\ dx=-du \end{bmatrix} = -\int u^{-1/3}du = -\frac{u^{-1/3+1}}{-1/3+1}+C$$

$$= -\frac{u^{2/3}}{2/3}+C$$

$$= -\frac{3}{2}u^{2/3}+C$$

$$= -\frac{3}{2}(1-x)^{2/3}+C$$

Therefore

$$\int_0^2 \frac{1}{\sqrt[3]{1-x}}dx = \int_0^1 \frac{1}{\sqrt[3]{1-x}}dx + \int_1^2 \frac{1}{\sqrt[3]{1-x}}dx$$

$$= \lim_{b\to 1^-}\int_0^b \frac{1}{\sqrt[3]{1-x}}dx + \lim_{a\to 1^+}\int_a^2 \frac{1}{\sqrt[3]{1-x}}dx$$

$$= \lim_{b\to 1^-}\left[-\frac{3}{2}(1-x)^{2/3}\right]_0^b + \lim_{a\to 1^+}\left[-\frac{3}{2}(1-x)^{2/3}\right]_a^2$$

$$= \lim_{b\to 1^-}\left(-\frac{3}{2}(1-b)^{2/3}+\frac{3}{2}(1-0)^{2/3}\right) + \lim_{a\to 1^+}\left(-\frac{3}{2}(1-2)^{2/3}+\frac{3}{2}(1-a)^{2/3}\right)$$

$$= \left(-\frac{3}{2}(1-1)^{2/3}+\frac{3}{2}(1-0)^{2/3}\right) + \left(-\frac{3}{2}(1-2)^{2/3}+\frac{3}{2}(1-1)^{2/3}\right)$$

$$= \left(0+\frac{3}{2}\cdot 1\right) + \left(-\frac{3}{2}\cdot 1+0\right) = \frac{3}{2} + \left(-\frac{3}{2}\right) = \boxed{0}$$

9. Use the Comparison Test to determine whether the integral $\int_2^\infty \frac{\cos^2 x}{x^2} dx$ converges or diverges. You do not need to evaluate the integral if it converges.

Solution: The integral $\int_2^\infty \frac{\cos^2 x}{x^2} dx$ is $\boxed{\text{convergent}}$, because $\frac{1}{x^2} \geq \frac{\cos^2 x}{x^2} \geq 0$ and $\int_1^\infty \frac{dx}{x^2}$ is convergent by the p-test, since $p = 2 > 1$.

10. Estimate the integral $\int_1^3 e^{x^2} dx$ using Simpson's Rule with $n = 4$ (*leave your answer in an e-form*). Give an upper bound for the error involved in this approximation.

Solution: Since

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

Simpson's Rule gives

$$\int_1^3 e^{x^2} dx \approx S_4 = \frac{\Delta x}{3}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)] = \boxed{\frac{1}{6}\left(e^{1^2} + 4e^{1.5^2} + 2e^{2^2} + 4e^{2.5^2} + e^{3^2}\right)}$$

To give an upper bound for the error involved in this approximation we note that

$$f'(x) = 2xe^{x^2}, \quad f''(x) = 2e^{x^2} + 4x^2e^{x^2}, \quad f'''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} = 12xe^{x^2} + 8x^3e^{x^2}$$

therefore

$$f^{(4)}(x) = 12e^{x^2} + 24x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2} = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} = 4e^{x^2}(3 + 12x^2 + 4x^4)$$

Since $4e^{x^2}(3 + 12x^2 + 4x^4)$ is an increasing function on $[1, 3]$, we have

$$|f^{(4)}(x)| = 4e^{x^2}(3 + 12x^2 + 4x^4) \leq 4e^{3^2}(3 + 12 \cdot 3^2 + 4 \cdot 3^4) = 1740e^9$$

Therefore

$$|E_S| \leq \frac{K_S(b-a)^5}{180n^4} = \frac{1740e^9(3-1)^5}{180(4)^4} = \boxed{\frac{29}{24}e^9}$$