Math 2413 Instructor: Adnan Said Chapter 1 Sections 1.2, 1.3, 1.4, and 1.5

1.2 Finding Limits Graphically and Numerically

Definition of a Limit

If f(x) becomes arbitrarily close to a single number L as x approaches c from the left, then $\lim_{x \to c^-} f(x) = L$.

If f(x) becomes arbitrarily close to a single number M as x approaches c from the right, then $\lim_{x \to c^+} f(x) = M$.

The limit exists if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

The limit fails to exist if any of the following occurs:

a) $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$.

b) f(x) increases or decreases without bound as x approaches c.

c) f(x) oscillates between two fixed values as x approaches c.

Create a table of values for the function and use the result to estimate the limit.

1) $f(x) = x^2 + 1$ 1) ____________

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	4.61	4.9601	4.9960		5.0040	5.0401	5.41

$\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 5$. So, $\lim_{x \to 2} f(x) = 5$

2) $f(x) = \dfrac{x+4}{x-2}$ 2) ____________

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	−59	−599	−5999		6001	601	61

$\lim_{x \to 2^-} f(x) = -\infty$ and $\lim_{x \to 2^+} f(x) = \infty$ So, $\lim_{x \to 2} f(x)$ DNE.

Complete the table and use the result to estimate the limit.

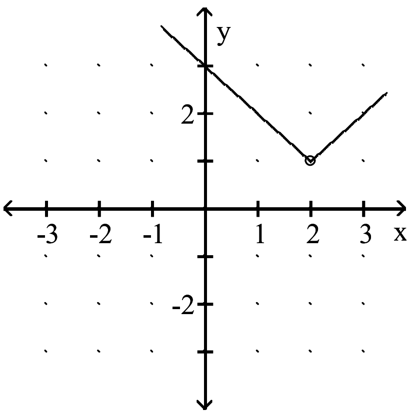
3) $f(x) = \sin\left(\frac{1}{x}\right)$ 3) ______________

x	0	$\frac{2}{11\pi}$	$\frac{2}{9\pi}$	$\frac{2}{7\pi}$	$\frac{2}{5\pi}$	$\frac{2}{3\pi}$	$\frac{2}{\pi}$
f(x)		−1	1	−1	1	−1	1

$\lim_{x \to 0^+} f(x)$ **DNE.**

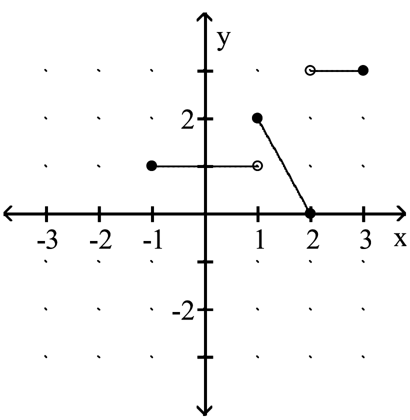
Solve the problem.

4) **Find the limit of f(x) as x approaches 2.** 4) ______________



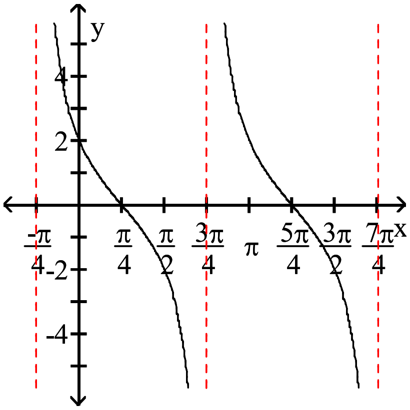
$\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 2^+} f(x) = 1$. So, $\lim_{x \to 2} f(x) = 1.$

5) **Find the limit of f(x) as x approaches 1.** 5) ______________



$\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = 2$ So, $\lim_{x \to 1} f(x)$ **DNE.**

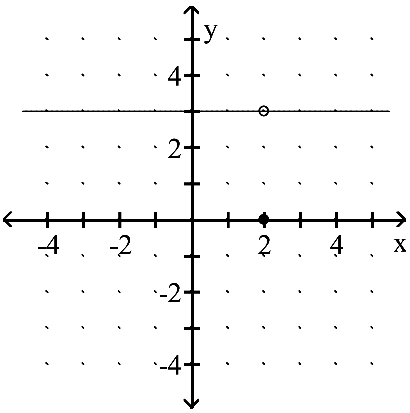
6) **Find the limit of f(x) as x approaches $\frac{3\pi}{4}$.** 6) ______________



$\lim\limits_{x \to \frac{3\pi}{4}^-} f(x) = -\infty$ and $\lim\limits_{x \to \frac{3\pi}{4}^+} f(x) = \infty$ So, $\lim\limits_{x \to \frac{3\pi}{4}} f(x)$ DNE.

Graph the function, and find the indicated limit (if it exists).

7) $f(x) = \begin{cases} 3, & x \neq 2 \\ 0, & x = 2 \end{cases}$ **Find the limit of f(x) as x approaches 2.** 7) ______________

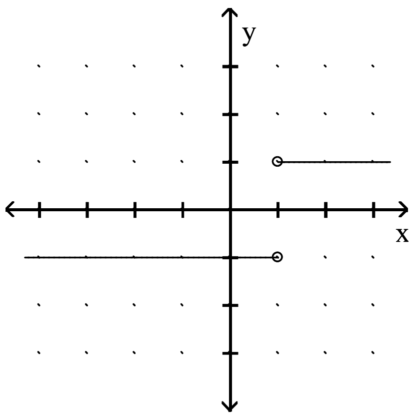


$\lim\limits_{x \to 2^-} f(x) = 3$ and $\lim\limits_{x \to 2^+} f(x) = 3$ So, $\lim\limits_{x \to 2} f(x) = 3.$

Graph the function, and find the indicated limit (if it exists) .

8) $f(x) = \dfrac{|x-1|}{x-1}$ **Find the limit of f(x) as x approaches 1.** 8) ____________

$\text{If } x > 1,\ f(x) = \dfrac{x-1}{x-1} = 1\ ;\ \text{If } x < 1,\ f(x) = \dfrac{-(x-1)}{x-1} = -1$



$\lim_{x \to 1^-} f(x) = -1$ and $\lim_{x \to 1^+} f(x) = 1$ So, $\lim_{x \to 1} f(x)$ DNE.

1.2 Exercises pg 55: (3, 13, 19, 20, 23) (17, 18, 25)

1.3 Evaluating Limits Analytically

Properties of Limits

1. $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x).$

2. $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x).$

3. $\lim_{x \to c} [b \cdot f(x)] = b \cdot \lim_{x \to c} f(x)$.

4. $\lim_{x \to c} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$.

5. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$

6. $\lim_{x \to c} b = b.$

Limit of a Polynomial Function

If f(x) is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.

Evaluate the limit if it exists.

9) $\lim_{x \to -2} (x^3 + 5x^2 - 7x + 1) = (-2)^3 + 5(-2)^2 - 7(-2) + 1 = 27$ 9) ____________

Limit of a Radical Function

If f(x) is a radical function, then $\lim_{x \to a} f(x) = f(a)$.

Evaluate the limit if it exists.

10) $\lim_{x \to -1} \sqrt{x^2 - 19x + 25} = \sqrt{(-1)^2 - 19(-1) + 25} = \sqrt{45} = 3\sqrt{5}$ 10) ____________

Limit of a Composite Function

If f(x) and g(x) are two functions, then $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$.

Evaluate the limit if it exists.

11) $\lim_{x \to 3} f(g(x))$ when $f(x) = x + 7$ and $g(x) = x^2$ 11) ____________

$\lim_{x \to 3} f(g(x)) = f\left(\lim_{x \to 3} g(x)\right) = f(9) = 16$

Limit of a Rational Function

If $f(x) = \dfrac{M(x)}{N(x)}$ is a rational function, the $\lim_{x \to a} f(x) = \dfrac{M(a)}{N(a)}$.

Case 1: If M(a) ≠ 0 and N(a) = 0, then the limit does not exist.

Evaluate the limit if it exists.

12) $\lim_{x \to 3} \dfrac{x^2 + 9}{x - 3} = \dfrac{3^2 + 9}{3 - 3} = \dfrac{18}{0}$. So, the limit DNE. 12) ____________

Case 2: If M(a) and N(a) are both zero, reduce the rational expression representing the function by factoring the numerator and denominator and cancelling out by a common factor.

Evaluate the limit if it exists.

13) $\lim_{x \to 2} \dfrac{x^3 + x^2 - 6x}{2x - 4} = \dfrac{2^3 + 2^2 - 6(2)}{2(2) - 4} = \dfrac{0}{0}$ 13) ____________

$$\frac{x^3 + x^2 - 6x}{2x - 4} = \frac{x(x^2 + x - 6)}{2(x - 2)} = \frac{x(x - 2)(x + 3)}{2(x - 2)} = \frac{x(x + 3)}{2}$$

So, $\lim_{x \to 2} \dfrac{x^3 + x^2 - 6x}{2x - 4} = \lim_{x \to 2} \dfrac{x(x + 3)}{2} = \dfrac{10}{2} = 5$

14) $\lim_{x\to 3} \frac{x^3-27}{4x-12} = \frac{3^3-27}{4(3)-12} = \frac{0}{0}$ 14) ____________

$$\frac{x^3-27}{4x-12} = \frac{(x-3)(x^2+3x+9)}{4(x-3)} = \frac{x^2+3x+9}{4}$$

So, $\lim_{x\to 3} \frac{x^3-27}{4x-12} = \lim_{x\to 3} \frac{x^2+3x+9}{4} = \frac{(3)^2+3(3)+9}{4} = \frac{27}{4}$

15) $\lim_{x\to 1} \frac{2x^4-2}{3x-3} = \frac{2(1)^4-2}{3(1)-3} = \frac{0}{0}$ 15) ____________

$$\frac{2x^4-2}{3x-3} = \frac{2(x^4-1)}{3(x-1)} = \frac{2(x^2-1)(x^2+1)}{3(x-1)} = \frac{2(x-1)(x+1)(x^2+1)}{3(x-1)}$$

$$= \frac{2(x+1)(x^2+1)}{3}$$

So, $\lim_{x\to 1} \frac{2x^4-2}{3x-3} = \lim_{x\to 1} \frac{2(x+1)(x^2+1)}{3} = \frac{2(2)(2)}{3} = \frac{8}{3}$

Case 3: If M(a) and N(a) are both zero, rationalize either the numerator or the denominator, then evaluate the limit.

Evaluate the limit if it exists.

16) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{0}{0}$ 16) ____________

$$\frac{(\sqrt{x+1}-1)}{x} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$$

$$= \frac{x+1+\sqrt{x+1}-\sqrt{x+1}-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{(\sqrt{x+1}+1)}.$$

So, $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{(\sqrt{x+1}+1)} = \frac{1}{2}$.

Case 4: If M(a) and N(a) are both zero, convert the complex fraction to a simple fraction , then evaluate the limit.

Evaluate the limit if it exists.

17) $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{2}}{x} = \dfrac{0}{0}$ 17) ____________

$$\frac{2(x+2) \cdot \dfrac{1}{x+2} - \dfrac{1}{2} \cdot 2(x+2)}{2(x+2) \cdot \dfrac{x}{1}} = \frac{2 - x - 2}{2x(x+2)} = \frac{-x}{2x(x+2)} = \frac{-1}{2(x+2)}.$$

So, $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{2}}{x} = \lim\limits_{x \to 0} \dfrac{-1}{2(x+2)} = -\dfrac{1}{4}$

Two Special Trigonometric Limits

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ **2.** $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Find the limit that involves a trigonometric function.

18) $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x} = \dfrac{0}{0}$ 18) ____________

$$= \lim_{x \to 0} (4)\left(\frac{\sin 4x}{4x}\right) = \lim_{x \to 0} (4) \cdot \lim_{x \to 0}\left(\frac{\sin 4x}{4x}\right) = (4)(1) = 4$$

19) $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{5x} = \dfrac{0}{0}$ 19) ____________

$$= \lim_{x \to 0}\left(\frac{-2}{5}\right) \cdot \left(\frac{1 - \cos 2x}{2x}\right) = \lim_{x \to 0}\left(\frac{-2}{5}\right) \cdot \lim_{x \to 0}\left(\frac{1 - \cos 2x}{2x}\right)$$

$$= \left(\frac{-2}{5}\right)(0) = 0$$

20) $\lim_{x \to 0} \frac{\tan x}{x} = \frac{0}{0}$ 20) ____________

$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right) = \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \lim_{x \to 0} \left(\frac{1}{\cos x}\right)$

$= (1)(1) = 1.$

1.3 Exercises pg 67:
(25, 35, 36, 44, 45, 63, 65, 67, 71) **(26, 53, 57, 64, 66, 72, 73)**

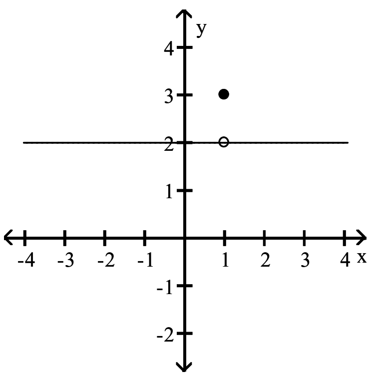
1.4 Continuity and One-Sided Limits

Continuity at a point : A function f is continuous at c if there are no holes, jumps, or gaps. A function f is continuous at c if the following three conditions are met:

1. f(c) is defined. **2.** $\lim_{x \to c} f(x)$ exists. **3.** $\lim_{x \to c} f(x) = f(c)$.

Find all points where the function is not continuous.

21) **f(x) is not continuous at x = 1.** 21) ____________



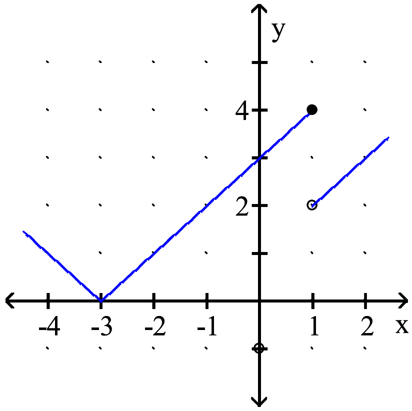
1. f(1) = 3, So f(1) is defined.

2. $\lim_{x \to 1^-} f(x) = 2$, $\lim_{x \to 1^+} f(x) = 2$ So, $\lim_{x \to 1} f(x) = 2$ (exists).

3. $\lim_{x \to 1} f(x) \neq f(1)$.

Find all points where the function is continuous.

22) **f(x) is not continuous at x = 1.** 22) ____________



1. $f(1) = 4$, So $f(1)$ is defined.
2. $\lim_{x\to 1^-} f(x) = 4$, $\lim_{x\to 1^+} f(x) = 2$ So, $\lim_{x\to 1} f(x)$ does not exist.

Find the constant a such that the function is continuous on the entire real line.

23) $f(x) = \begin{cases} 3x^3, & x \le 1 \\ ax + 5, & x > 1 \end{cases}$ 23) ____________

f(x) has to be continuous at x = 1, So, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

Then, $a + 5 = 3 \rightarrow$ **$a = -2$**

Discontinuities fall in two categories: removable and nonremovable.

Removable Discontinuity

A discontinuity at c is called removable if f can be made continuous by appropriately redefining the function f(x).

Discuss the continuity of each function at the indicated value.

24) $f(x) = \dfrac{3}{x+2}$, at $x = -2$ 24) ____________

f(x) is not defined at x = −2. There is no way to redefine f(x), in order to make the function continuous at x = −2.

So, f(x) has a nonremovable discontinuity at x = −2.

25) **$f(x) = \frac{x^2 - 1}{4x - 4}$, at $x = 1$** 25) ____________

f(x) is not defined at x = 1.

f(x) can be redefined as $f(x) = \frac{(x+1)(x-1)}{4(x-1)} = 4(x+1)$.

Now the function is continuous at x = 1.

So, f(x) has a removable discontinuity at x = 1.

26) **$f(x) = \begin{cases} x^2 + 1, & x > 0 \\ x + 1, & x \le 0 \end{cases}$, at $x = 0$** 26) ____________

1. f(0) = 1, So f(0) is defined.
2. $\lim_{x \to 0^+} f(x) = 1$, $\lim_{x \to 0^-} f(x) = 1$ Then, $\lim_{x \to 0} f(x) = 1$(exists).
3. So, $\lim_{x \to 0} f(x) = f(0)$

So, f(x) is continuous at x = 0.

To find the one–sided limit of a function f(x) algebraically, use the rule: $\lim_{x \to a^{\pm}} f(x) = f(a)$.

If a function is not defined for values required to evaluate the right–hand or left–hand limit, then the limit does not exist.

Find the limit , if it exists.

27) $\lim_{x \to 3^-} (x^3 + \sqrt{9 - x^2}) = \lim_{x \to 3^-} x^3 + \lim_{x \to 3^-} \sqrt{9 - x^2} = 27 + 0 = 27$ 27) ____________

28) $\lim_{x \to 3^+} \sqrt{9 - x^2}$ **; f(x) is not defined for x > 3, the limit DNE.** 28) ____________

29) $\lim_{x \to 2^+} \frac{4x - 8}{x^2 - 4} = \lim_{x \to 2^+} \frac{4(x-2)}{(x+2)(x-2)} = \lim_{x \to 2^+} \frac{4}{x+2} = \frac{4}{4} = 1$ 29) ____________

30) $\lim_{x \to -1^-} f(x) = \begin{cases} 5x + 2, & x < -1 \\ x - 2, & x > -1 \end{cases}$ 30) ____________

$\lim_{x \to -1^-} f(x) = 5(-1) + 2 = -5 + 2 = -3$

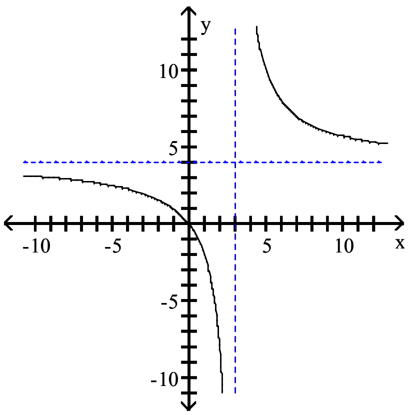
1.4 Exercises pg79
(3, 5, 9, 18, 40, 47, 62, 77) **(11, 14, 41, 54, 80)**

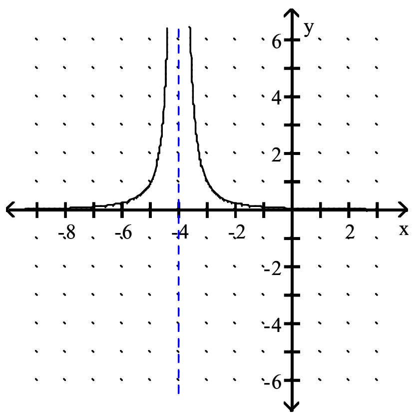
1.5 Infinite limits

Infinite Limit: A limit in which f(x) increases or decreases without bound as x approaches c is called an infinite limit.

Determine the limit of the function.

31) **a) as $x \to 3^-$ and as $x \to 3^+$** **b) as $x \to -4^-$ and as $x \to -4^+$** 31) ____________





$\lim_{x \to 3^-} f(x) = -\infty$

$\lim_{x \to 3^+} f(x) = \infty$

$\lim_{x \to -4^-} f(x) = \infty$

$\lim_{x \to -4^+} f(x) = \infty$

Vertical Asymptote

If f(x) approaches infinity (or negative infinit) as x approaches c from the left or the right, then the line x = c is a vertical asymptote of the graph of f(x). The vertical asymptote occurs at a value at which the denominator is zero.

Find any vertical asymptotes.

32) $f(x) = \dfrac{1}{2(x+1)}$ 32) ____________

When $x = -1$, the denominator is zero and the numerator is not zero. $x = -1$ is a vertical asymptote of the graph of f(x).

33) $f(x) = \dfrac{x^2+9}{x^2-9} = \dfrac{x^2+9}{(x-3)(x+3)}$ 33) ____________

When $x = 3$ and $x = -3$, the denominator is zero and the numerator is not zero. $x = 3$ and $x = -3$ are vertical asymptotes of the graph of f(x).

34) $f(x) = \dfrac{x^2+2x-8}{x^2-4} = \dfrac{(x-2)(x+4)}{(x-2)(x+2)}$ 34) ____________

When $x = 2$, the denominator is zero and the numerator is zero. So, $x = 2$ is not a vertical asymptote of the graph of f(x).
When $x = -2$, the denominator is zero and the numerator is not zero. $x = -2$ is a vertical asymptote of the graph of f(x).

Find each limit .

35) $\lim\limits_{x \to 1^+} \dfrac{x^2+3x}{x-1}$ 35) ____________

When $x = 1$, the denominator is zero and the numerator is not zero. Then $x = 1$ is a vertical asymptote of the graph of f(x). This means that the limit is either ∞ or $-\infty$. The result can be determined by analyzing f at values of x close to 1.

For example, f(2) is positive, then $\lim\limits_{x \to 1^+} \dfrac{x^2+3x}{x-1} = \infty$.

36) $\lim_{x \to -1^-} \frac{x^2 - 2x + 1}{x + 1}$ 36) ____________

When x = −1, the denominator is zero and the numerator is not zero. Then x = −1 is a vertical asymptote of the graph of f(x).

This means that the limit is either ∞ or $-\infty$. The result can be determined by analyzing f at values of x close to −1.

For example, f(−2) is negative, then $\lim_{x \to -1^-} \frac{x^2 - 2x + 1}{x + 1} = -\infty$.

37) $\lim_{x \to 5^+} \left(x^2 + \frac{3}{x - 5}\right) = \lim_{x \to 5^+} x^2 + \lim_{x \to 5^+} \frac{3}{x - 5}$ 37) ____________

When x = 5, the denominator is zero and the numerator is not zero. Then x = 5 is a vertical asymptote of the graph of f(x). This means that the limit is either ∞ or $-\infty$. The result can be determined by analyzing f at values of x close to 5. For example, f(6) = 3(positive), then $\lim_{x \to 5^+} \frac{3}{x - 5} = \infty$.

So, $\lim_{x \to 5^+} \left(x^2 + \frac{3}{x - 5}\right) = 25 + \infty = \infty$

38) $\lim\limits_{x \to \frac{\pi}{2}^+} \frac{-2}{\cos x}$ 38) ____________

When $x = \frac{\pi}{2}$, the denominator is zero and the numerator is not zero. Then $x = \frac{\pi}{2}$ is a vertical asymptote of the graph of f(x).

This means that the limit is either ∞ or $-\infty$. The result can be determined by analyzing f at values of x close to $\frac{\pi}{2}$.

For example, $f(\pi)$ is positive, then $\lim\limits_{x \to \frac{\pi}{2}^+} \frac{-2}{\cos x} = \infty$

1.5 Exercises pg88
(2, 3, 15, 17, 29, 33, 37, 41, 43) **(22, 24, 30, 34, 38, 42)**