3.1 Extrema of a Function on an Interval

Definition of Extrema

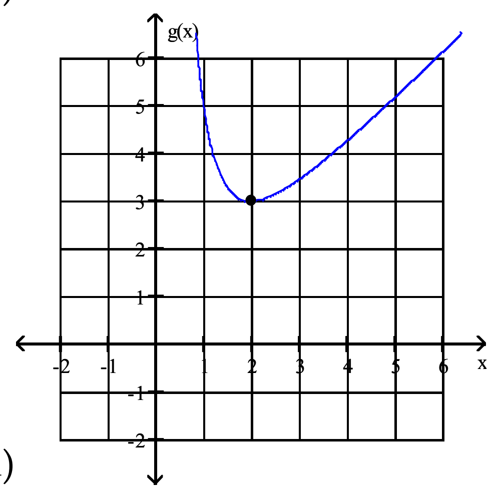
Let f be defined on an interval I containing c.

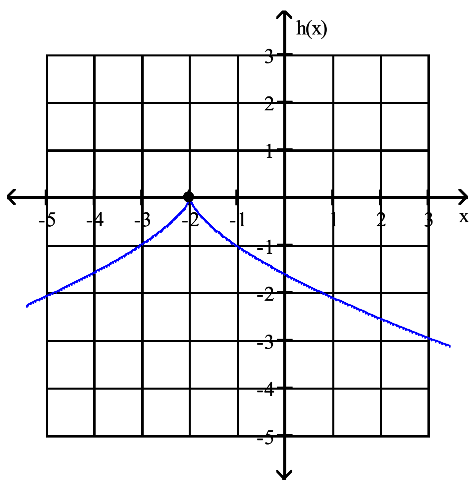
1. f(c) is the **maximum** of on I when f(c) ≥ f(x) for all x in the interval I.
2. f(c) is the **minimum** of on I when f(c) ≤ f(x) for all x in the interval I.

The maximum and minimum of a function on an interval are the extreme values, or **extrema** of the function on the interval. The extreme values are also called the **absolute maximum** and the **absolute minimum** on the interval.

Find the location of the absolute extrema for the function.

1) a) b) 1) ____________





a)
Absolute maximum: None
Absolute minimum: 3 at x = 2.

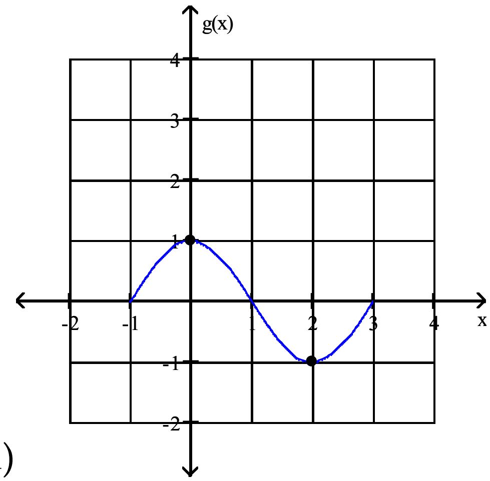
b)
Absolute maximum: 0 at x = −2
Absolute minimum: None.

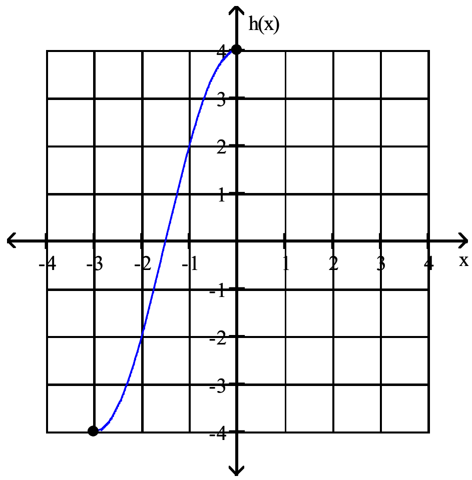
The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Find the location of the absolute extrema for the function.

2) a) b) 2) ____________





a)

Absolute maximum: 1 at x = 0.

Absolute minimum: −1 at x = 2.

b)

Absolute maximum: 4 at x = 0.

Absolute minimum: −4 at x = −3.

Definition of Relative Extrema

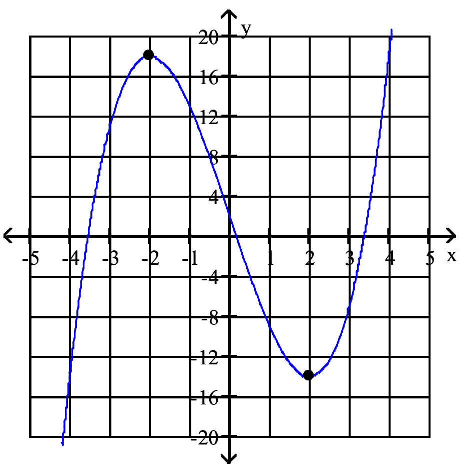
The graph of a typical function may have "peaks" or "valleys". Peaks and valleys are sometimes called turning points of the graph, because as you move from left to right, the graph changes from increasing to decreasing at a peak and from decreasing to increasing at a valley. **Relative maximum** and **relative minimum** are also called **local maximum** and **local minimum**, respectively.

Definition of Critical Numbers

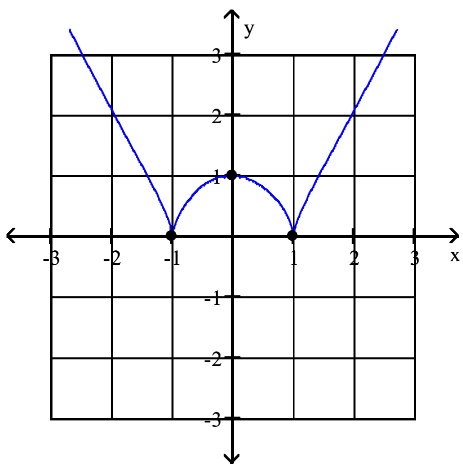
At each relative maximum and relative minimum, the derivative is either zero or does not exist. The x–values at these special points are called **critical numbers**.

Find the location of the relative extrema, and the critical numbers for the function.

3) a.



b.



3) ______________

a. Relative maximum: (−2, 18).
   Relative minimum: (2, −14).
   Critical numbers: x = −2 and x = 2

b. Relative maximum: (0, 1).
   Relative minimum: (−1, 0) and (1, 0).
   Critical numbers: x = −1, x = 0, and x = 1

Find the critical numbers for the function.

4) **$f(x) = 2\sin x - \cos 2x$ on the interval $[0, 2\pi]$.** 4) ____________

$f'(x) = 2\cos x - (-2\sin 2x) = 2\cos x + 2\sin 2x$

$= 2\cos x + 2(2\sin x\cos x) = 2\cos x + 4\sin x\cos x$

$= 2\cos x(1 + 2\sin x)$

$f'(x) = 0 \to 2\cos x(1 + 2\sin x) = 0$

$\to 2\cos x = 0$ or $1 + 2\sin x = 0 \to \cos x = 0$ or $\sin x = -\dfrac{1}{2}$

$\to x = \dfrac{\pi}{2}, \; x = \dfrac{3\pi}{2}$ or $x = \dfrac{7\pi}{6}, \; x = \dfrac{11\pi}{6}$

Because $f'(x)$ is defined for all x, then there are no x−values in which $f'(x)$ does not exist.

So, the only critical numbers of f(x) in $(0, 2\pi)$ are:

$x = \dfrac{\pi}{2}, \; x = \dfrac{3\pi}{2}, \; x = \dfrac{7\pi}{6}, \; x = \dfrac{11\pi}{6}$

Guidelines for Finding Extrema on a Closed Interval

1. Find the critical numbers of f(x).
2. Evaluate f(x) at each critical number in (a, b).
3. Evaluate f(x) at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest value is the maximum.

Find the location of the extrema of the function.

5) **$f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.** 5) ____________

1. $f'(x) = 12x^3 - 12x^2$

$f'(x) = 0 \to 12x^3 - 12x^2 = 0$

$\to 12x^2(x - 1) = 0 \to x = 0$ or $x = 1$

Because $f'(x)$ is defined for all x, then there are no x−values in which $f'(x)$ does not exist.

The only critical numbers of f(x) in $(-1, 2)$ are: $x = 0$ and $x = 1$.

The minimum or maximum will occur at the critical numbers or at the endpoints of the interval $[-1, 2]$.

x	−1	0	1	2
f(x)	7	0	-1	16

The maximum is 16 at $x = 2$, and the minimum is -1 at $x = 1$.

Find the location of the extrema of the function.

6) **$f(x) = 2x - 3x^{2/3}$ on the interval $[-1, 3]$.** 6) ____________

$f'(x) = 2 - 2x^{-1/3} = 2 - \frac{2}{x^{1/3}} = \frac{2x^{1/3}}{x^{1/3}} - \frac{2}{x^{1/3}} = \frac{2x^{1/3} - 2}{x^{1/3}}$

$f'(x) = 0 \rightarrow 2x^{1/3} - 2 = 0 \rightarrow x^{1/3} = 1 \rightarrow x = 1$

$f'(x)$ is not defined at $x = 0$, then $f'(x)$ does not exist at $x = 0$.
The critical numbers of $f(x)$ in $(-1, 3)$ are: $x = 0$ and $x = 1$.
The minimum or maximum will occur at the critical numbers or at the endpoints of the interval $[-1, 3]$.

x	−1	0	1	3
f(x)	−5	0	−1	- 0.24

The maximum is 0 at $x = 0$, and the minimum is -5 at $x = -1$.

Find the location of the extrema of the function.

7) **$f(x) = \frac{1}{2}x - \sin x$ in the interval $[0, 2\pi]$.** 7) ____________

$f'(x) = \frac{1}{2} - \cos x$

$f'(x) = 0 \rightarrow \frac{1}{2} - \cos x = 0 \rightarrow \cos x = \frac{1}{2} \rightarrow x = \frac{\pi}{3}, x = \frac{5\pi}{3}$

Because $f'(x)$ is defined for all x, then there are no x−values in which $f'(x)$ does not exist. So, the only critical numbers of $f(x)$ in $(0, 2\pi)$ are: $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$.

x	0	$\frac{\pi}{3}$	$\frac{5\pi}{3}$	2π
f(x)	0	$\frac{\pi - 3\sqrt{3}}{6}$	$\frac{5\pi + 3\sqrt{3}}{6}$	π

The maximum is $\frac{5\pi + 3\sqrt{3}}{6}$ at $x = \frac{5\pi}{3}$.

The minimum is $\frac{\pi - 3\sqrt{3}}{6}$ at $x = \frac{\pi}{3}$.

3.1 Exercises pg 167 (9, 11, 19, 23, 33) **(10, 15, 21, 25, 35)**

3.2 Rolle's Theorem

Rolle's Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$ then there is at least one number c in (a, b) such that $f'(c) = 0$.

Determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$.
If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$.

8) **$f(x) = x^4 - 2x^2$, $[-2, 2]$** 8) ____________

f is continuous on the closed interval $[-2, 2]$.

$f'(x) = 4x^3 - 4x$

f is differentiable on the open interval $(-2, 2)$.

$f(-2) = 16 - 8 = 8$; $f(2) = 16 - 8 = 8$ $\rightarrow$ $f(-2) = f(2)$

Rolle's Theorem can be applied to f on the closed interval $[-2, 2]$.

$f'(x) = 0 \rightarrow 4x^3 - 4x = 0$

$\rightarrow 4x(x^2 - 1) = 0 \rightarrow 4x(x - 1)(x + 1) = 0$

$\rightarrow x = 0$ or $x = 1$ or $x = -1$

In the interval $(-2, 2)$, $f'(x) = 0$ at $x = 0$ or $x = 1$ or $x = -1$

9) **$f(x) = \dfrac{x^2}{1 - x^2}$, $[-2, 2]$** 9) ____________

f(1) does not exist , then f is not continuous on the closed interval $[-2, 2]$. Rolle's Theorem can not be applied to f on the closed interval $[-2, 2]$.

10) **$f(x) = \csc x$, $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{2}\right]$** 10) ____________

$f(\pi)$ does not exist, then f is not continuous on the closed interval $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{2}\right]$. Rolle's Theorem can not be applied to f on the closed interval $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{2}\right]$.

11) **$f(x) = \sin 2x, \left[0, \frac{\pi}{3}\right]$** 11) ____________

f is continuous on the closed interval $\left[0, \frac{\pi}{3}\right]$.

$f'(x) = 2\cos 2x$

f is differentiable on the open interval $\left(0, \frac{\pi}{3}\right)$.

$f(0) = \sin 0 = 0$; $f\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $\rightarrow$ $f(0) \neq f\left(\frac{\pi}{3}\right)$. Rolle's Theorem can not be applied to f on the closed interval $\left[0, \frac{\pi}{3}\right]$.

12) **$f(x) = 2x + 4\cos^2 x, [0, \pi]$** 12) ____________

f is continuous on the closed interval $[0, \pi]$.

$f(x) = 2x + 4(\cos x)^2$

$f'(x) = 2 - 8\cos x \sin x = 2 - 4(2\cos x \sin x) = 2 - 4\sin 2x$

f is differentiable on the open interval $(0, \pi)$.

$f(0) = 4$; $f(\pi) = 2\pi + 4 \rightarrow$ $f(0) \neq f(\pi)$ Rolle's Theorem can not be applied to f on the closed interval $[0, \pi]$.

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

13) **$f(x) = 5 - \dfrac{4}{x}$ on the interval $[1, 4]$.** 13) ____________

f is continuous on the closed interval $[1, 4]$.

$f(x) = 5 - \dfrac{4}{x} = 5 - 4x^{-1} \rightarrow f'(x) = 4x^{-2} = \dfrac{4}{x^2}$

f is differentiable on the open interval $(1, 4)$. The Mean Value Theorem can be applied. So, there exists at least one number x in $(1, 4)$ such that: $f'(x) = \dfrac{f(4) - f(1)}{4 - 1}$

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1$$

Set $f'(x) = 1 \rightarrow \dfrac{4}{x^2} = 1 \rightarrow x^2 = 4 \rightarrow x = 2$ or $x = -2$

In the interval $(1, 4)$, $x = 2$.

14) **$f(x) = \sin x + \cot x$ on the interval $[0, 2\pi]$.** 14) ____________

$f(\pi)$ does not exist , then f is not continuous on the closed interval $[0, 2\pi]$. The Mean Value Theorem does not apply.

3.2 Exercises pg 174

(9, 13, 17, 21, 39, 41, 45) (14, 16, 18, 22, 40, 42, 46)